On the Cohomology of the Invariant Euler–Lagrange Complex

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Abstract

Given a Lie group action G we show, using the method of equivariant moving frames, that the local cohomology of the invariant Euler–Lagrange complex is isomorphic to the Lie algebra cohomology of G.

1 Introduction

The *variational bicomplex* is a double complex of differential forms defined on the infinite extended jet bundle $\mathrm{J}^\infty(M,p)$ of p-dimensional submanifolds of a manifold M. It provides a natural and general differential geometric framework for variational calculus. The modern form of the theory originates from Vinogradov's, [33, 34, 35], and Tulczyjew's, [32], work. The later contributions of Anderson, [1, 2], have demonstrated the power and efficacy of the bicomplex formalism for both local and global problems in the calculus of variations. The variational bicomplex is an important theoretical tool for studying the geometry of differential equations, [31]. It is used to compute geometric and topological quantities of interest, including characteristic cohomology, [8, 9], characteristic classes, [1], Helmholtz conditions, [1], conservation laws, [3, 4], and null Lagrangians, [23].

Of particular interest is the complex associated with the edge of the augmented variational bicomplex. The Euler operator or variational derivative is intrinsically defined as the corner map of this edge complex and for this reason it is called the *Euler–Lagrange complex*. This complex provides tools for studying many problems in the calculus of variations. In the presence of a Lie group action it is natural to investigate invariant problems in the calculus of variations; to this end it is useful to study the G-invariant variational bicomplex and its cohomology, [1, 2, 5, 6, 20]. For Lie groups acting projectably on fiber bundles, Anderson and Pohjanpelto have shown that the local cohomology of the G-invariant Euler–Lagrange complex is isomorphic to the Lie

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algebra cohomology of G, [5]. An important feature of their proof is that it is constructive and readily lends itself to studying particular examples. Recently, Itskov, [16, 17], proved, using arguments from $\mathcal{C}$-spectral theory, that the isomorphism still holds for non-projectable group actions. A drawback of Itskov's proof is that it is difficult to apply in particular examples. One purpose of this paper is to give a simplified and constructive proof of his theorem which can easily be applied to particular problems. The construction of the isomorphism is completely algorithmic and can in principle be implemented in symbolic software packages such as MATHEMATICA or MAPLE.

The proofs found in this paper are natural extensions of the original proofs invented by Anderson and Pohjanpelto, [1, 2, 5]. A novel feature is the incorporation of the *equivariant moving frame* method developed by Fels and Olver, [13, 14], into the constructions. For a general finite-dimensional transformation group G, a moving frame is defined as an equivariant map from an open subset of the jet space of submanifolds to the Lie group G. Once a moving frame is established, it provides a canonical mechanism, called *invariantization*, of associating an invariant differential jet form to an arbitrary differential jet form. The *G-invariant variational complex* is obtained in essence by applying invariantization to the free variational bicomplex. The theoretical foundations of this construction appear in the work of Kogan and Olver, [19, 20], where the authors establish a general formula relating invariant variational problems to their invariant Euler–Lagrange equations. For non-projectable group actions, a key observation is that the resulting invariant complex relies on three differentials with nonstandard commutation relations (and so is no longer a bicomplex in the usual form).

The structure of the paper is as follows. In Section 2 we recall some standard facts about the free variational bicomplex and its cohomology. Sections 3 and 4 contain an overview of the moving frame construction and the invariantization of the free variational bicomplex. The main results of the paper appear in Sections 5 and 6. By introducing an invariant connection on the invariant horizontal total differential operators we show that the interior rows of the invariant variational bicomplex are locally exact. From this it follows that the cohomology of the invariant Euler–Lagrange complex $H^*(\widetilde{\mathcal{E}}_G)$ is locally isomorphic to the de Rham cohomology $H^*(\mathbf{\Omega}^*_G)$ of invariant differential forms on $\mathrm{J}^\infty(M,p)$. The moving frame associated to the group action G gives an immediate local isomorphism between the Lie algebra cohomology $H^*(\mathfrak{g}^*)$ and the de Rham cohomology $H^*(\mathbf{\Omega}^*_G)$ from which we conclude that $H^*(\mathfrak{g}^*) \simeq H^*(\widetilde{\mathcal{E}}_G)$. The theory is illustrated by three examples in Section 8: the actions of the special Euclidean and special affine groups on curves in the plane and the action of the special Euclidean group on surfaces.

2 The Variational Bicomplex

We begin with a brief review of the variational bicomplex. We refer the reader to [1, 2, 18, 31] for a detailed exposition. Basic results on jet bundles, contact forms, et cetera can be found in [23, 24, 35, 36].

Let M be a smooth m-dimensional manifold. We denote by $\mathrm{J}^n = \mathrm{J}^n(M,p)$ the n^{th} order *extended jet bundle* of equivalence classes of p-dimensional submanifolds $S \subset M$ under the equivalence relation of n^{th} order contact, where $0 < p < m$. The infinite jet bundle $\mathrm{J}^\infty = \mathrm{J}^\infty(M,p)$ is defined as the inverse limit of the finite order jet bundles under the standard projections $\pi^{n+1}_n \colon \mathrm{J}^{n+1} \to \mathrm{J}^n$. Differential functions and differential forms on J^n will be identified with their pull-backs to the appropriate open subset of J^∞.

Locally we can identify $M \simeq X \times U$ with the cartesian product of the submanifolds X and U with local coordinates $x = (x^1, \ldots, x^p)$ and $u = (u^1, \ldots, u^q)$ respectively. The coordinates on X are considered as independent variables while the coordinates on U are considered as dependent variables. This induces local coordinates $z^{(\infty)} = (x, u^{(\infty)})$ on J^∞, where $u^{(\infty)}$ denotes the collection of derivatives u^α_J, $\alpha = 1, \ldots, q$, $\#J \geq 0$, of arbitrary order. Here $J = (j_1, \ldots, j_k)$, with $1 \leq j_\nu \leq p$, is a symmetric multi-index of order $k = \#J$. Coordinates $z^{(n)} = (x, u^{(n)})$ on the jet bundle J^n are obtained by truncating $z^{(\infty)}$ at order n.

Definition 2.1. A differential form θ on J^∞ is called a *contact form* if it is annihilated by all submanifold jets, that is, $\theta|_{j_\infty S} = 0$ for every p-dimensional submanifold $S \subset M$.

The subbundle of the cotangent bundle $T^*\mathrm{J}^\infty$ spanned by the contact one-forms is called the *contact* or *vertical subbundle* and denoted by $\mathcal{C}^{(\infty)}$. In the local coordinates $(x, u^{(\infty)})$, every contact one-form is a linear combination of the *basic contact one-forms*

$$\theta^\alpha_J = du^\alpha_J - \sum_{i=1}^{p} u^\alpha_{J,i} dx^i, \qquad \alpha = 1, \ldots, q, \qquad \#J \geq 0. \tag{2.1}$$

On the other hand, the one-forms

$$dx^i, \qquad i = 1, \ldots, p, \tag{2.2}$$

span the *horizontal subbundle*, denoted by $\mathbf{H}^*$. This induces a local splitting $T^*\mathrm{J}^\infty = \mathbf{H}^* \oplus \mathcal{C}^{(\infty)}$ of the cotangent bundle. Note that this splitting depends of course on the chosen coordinates. Any one-form Ω on J^∞ can be uniquely decomposed into horizontal and vertical components, $\Omega = \pi_H(\Omega) + \pi_V(\Omega)$, where $\pi_H \colon T^*\mathrm{J}^\infty \to \mathbf{H}^*$ and $\pi_V \colon T^*\mathrm{J}^\infty \to \mathcal{C}^{(\infty)}$ are the induced horizontal and vertical (or contact) projections.

The splitting of $T^*\mathrm{J}^\infty$ induces a bigrading of the differential forms on J^∞. The space of differential forms of horizontal degree r and vertical degree s is denoted by $\mathbf{\Omega}^{r,s} = \mathbf{\Omega}^{r,s}(\mathrm{J}^\infty)$. Then

$$\mathbf{\Omega}^*(\mathrm{J}^\infty) = \mathbf{\Omega}^* = \bigoplus_{r,s=0}^{\infty} \mathbf{\Omega}^{r,s}. \tag{2.3}$$

Under the bigrading (2.3), the differential d on J^∞ splits into horizontal and vertical components, $d = d_H + d_V$, where d_H increases horizontal degree and d_V increases vertical degree. Closure, $d^2 = d \circ d = 0$, implies

$$d_H^2 = 0, \qquad d_H \circ d_V + d_V \circ d_H = 0, \qquad d_V^2 = 0. \tag{2.4}$$

The *horizontal differential* of a differential function F is the horizontal one-form

$$d_H F = \sum_{i=1}^{p} (D_i F) dx^i, \qquad \text{where} \qquad D_i = \frac{\partial}{\partial x^i} + \sum_{\alpha=1}^{q} \sum_J u^\alpha_{J,i} \frac{\partial}{\partial u^\alpha_J} \tag{2.5}$$

denotes the usual total derivative with respect to x^i. The *vertical differential* of a differential function F is the contact one-form

$$d_V F = \sum_{\alpha=1}^{q} \sum_J \frac{\partial F}{\partial u^\alpha_J} \theta^\alpha_J. \tag{2.6}$$

To obtain the full variational bicomplex we append to each row a certain quotient space of the differential forms of maximal horizontal degree. Define the quotient and standard quotient projections[1]

$$\mathcal{F}^s = \boldsymbol{\Omega}^{p,s}/d_H(\boldsymbol{\Omega}^{p-1,s}), \qquad \pi\colon \boldsymbol{\Omega}^{p,s} \to \mathcal{F}^s, \qquad s \geq 1.$$

The spaces $\mathcal{F}^s$ are called the spaces of type s *functional forms* on J^∞. The quotient projection plays the role of an integration by parts operator and is essential to the derivation of the Euler–Lagrange equations using the variational bicomplex formalism. By virtue of (2.4), the composition

$$\delta_V = \pi \circ d_V \tag{2.7a}$$

is a boundary operator from $\mathcal{F}^s$ to $\mathcal{F}^{s+1}$. Finally the *Euler operator* is defined as

$$E = \pi \circ d_V \colon \boldsymbol{\Omega}^{p,0} \to \mathcal{F}^1. \tag{2.7b}$$

Definition 2.2. The (augmented) variational bicomplex is the double complex ($\boldsymbol{\Omega}^{*,*}$, d_H, d_V) of differential forms on the infinite jet bundle J^∞:

$$\begin{tikzcd}
 & & \vdots & \vdots & & \vdots & \vdots & \vdots & \\
 & 0 \arrow[r] & \boldsymbol{\Omega}^{0,3} \arrow[u, "d_V"] \arrow[r, "d_H"] & \boldsymbol{\Omega}^{1,3} \arrow[u, "d_V"] \arrow[r, "d_H"] & \cdots \arrow[r, "d_H"] & \boldsymbol{\Omega}^{p-1,3} \arrow[u, "d_V"] \arrow[r, "d_H"] & \boldsymbol{\Omega}^{p,3} \arrow[u, "d_V"] \arrow[r, "\pi"] & \mathcal{F}^3 \arrow[u, "\delta_V"] \arrow[r] & 0 \\
 & 0 \arrow[r] & \boldsymbol{\Omega}^{0,2} \arrow[u, "d_V"] \arrow[r, "d_H"] & \boldsymbol{\Omega}^{1,2} \arrow[u, "d_V"] \arrow[r, "d_H"] & \cdots \arrow[r, "d_H"] & \boldsymbol{\Omega}^{p-1,2} \arrow[u, "d_V"] \arrow[r, "d_H"] & \boldsymbol{\Omega}^{p,2} \arrow[u, "d_V"] \arrow[r, "\pi"] & \mathcal{F}^2 \arrow[u, "\delta_V"] \arrow[r] & 0 \\
 & 0 \arrow[r] & \boldsymbol{\Omega}^{0,1} \arrow[u, "d_V"] \arrow[r, "d_H"] & \boldsymbol{\Omega}^{1,1} \arrow[u, "d_V"] \arrow[r, "d_H"] & \cdots \arrow[r, "d_H"] & \boldsymbol{\Omega}^{p-1,1} \arrow[u, "d_V"] \arrow[r, "d_H"] & \boldsymbol{\Omega}^{p,1} \arrow[u, "d_V"] \arrow[r, "\pi"] & \mathcal{F}^1 \arrow[u, "\delta_V"] \arrow[r] & 0 \\
0 \arrow[r] & \mathbb{R} \arrow[r] & \boldsymbol{\Omega}^{0,0} \arrow[u, "d_V"] \arrow[r, "d_H"] & \boldsymbol{\Omega}^{1,0} \arrow[u, "d_V"] \arrow[r, "d_H"] & \cdots \arrow[r, "d_H"] & \boldsymbol{\Omega}^{p-1,0} \arrow[u, "d_V"] \arrow[r, "d_H"] & \boldsymbol{\Omega}^{p,0} \arrow[u, "d_V"] \arrow[ur, "E"'] & &
\end{tikzcd}$$

The following theorems summarize the local theory of the variational bicomplex.

Theorem 2.3. For each $r = 0, 1, 2, \ldots, p$, the vertical complex

$$0 \longrightarrow \boldsymbol{\Omega}^r_X \xrightarrow{(\pi^\infty_X)^*} \boldsymbol{\Omega}^{r,0} \xrightarrow{d_V} \boldsymbol{\Omega}^{r,1} \xrightarrow{d_V} \boldsymbol{\Omega}^{r,2} \xrightarrow{d_V} \cdots$$

is locally exact. Here $\boldsymbol{\Omega}^r_X$ is the space of r forms over X and $\pi^\infty_X\colon \mathrm{J}^\infty \to X$ is the projection onto the space of independent variables induced by a choice of local coordinates $M \simeq X \times U$ on the manifold M.

The proof is similar to the proof of the Poincaré lemma for the de Rham complex, [7, 23].

[1]This is one of two equivalent approaches. Alternatively the *interior Euler operators* $I\colon \boldsymbol{\Omega}^{p,s} \to \boldsymbol{\Omega}^{p,s}$ may be introduced and the images $I(\boldsymbol{\Omega}^{p,s})$ used instead of the spaces $\mathcal{F}^s$. Both viewpoints will be used in the sequel.

Theorem 2.4. For each $s \geq 1$, the augmented horizontal complex

$$0 \longrightarrow \mathbf{\Omega}^{0,s} \xrightarrow{d_H} \mathbf{\Omega}^{1,s} \xrightarrow{d_H} \cdots \xrightarrow{d_H} \mathbf{\Omega}^{p,s} \xrightarrow{\pi} \mathcal{F}^s \longrightarrow 0$$

is locally exact.

One method of proof consists of verifying that

$$h^{r,s}(\omega) = \frac{1}{s} \sum_{\#I=0}^{k-1} \sum_{j=1}^{p} \sum_{\alpha=1}^{q} \frac{\#I+1}{p-r+\#I+1} D_I[\theta^\alpha \wedge F_\alpha^{I,j}(\omega_j)], \tag{2.8}$$

where $\omega_j = D_j \lrcorner\, \omega$ denotes the interior product of ω with D_j, k is the order of ω and

$$F_\alpha^I(\omega) = \sum_{\#J=0}^{k-\#I} \binom{\#I+\#J}{\#J} (-D)_J \left(\frac{\partial}{\partial u_{I,J}^\alpha} \lrcorner\, \omega \right) \tag{2.9}$$

are the *interior Euler operators*, are local horizontal homotopy operators, [1].

Theorem 2.5. The Euler–Lagrange complex $\mathcal{E}^*(\mathrm{J}^\infty)$

$$0 \longrightarrow \mathbb{R} \longrightarrow \mathbf{\Omega}^{0,0} \xrightarrow{d_H} \mathbf{\Omega}^{1,0} \xrightarrow{d_H} \cdots \xrightarrow{d_H} \mathbf{\Omega}^{p,0} \xrightarrow{E} \mathcal{F}^1 \xrightarrow{\delta_V} \mathcal{F}^2 \xrightarrow{\delta_V} \cdots$$

is locally exact.

This result may be established using Theorems 2.3 and 2.4 and homological algebra arguments. Alternatively, one may construct explicit homotopy operators, [1, 23]. There is also a global version of Theorem 2.5, giving an isomorphism of the cohomology of $\mathcal{E}^*(\mathrm{J}^\infty)$ with the de Rham cohomology of J^∞, [1].

3 Moving Frames

There are now a wide variety of papers on the theory of equivariant moving frames, [13, 14, 20, 27]. In this section we recall the results relevant to our problem.

Let G be an r-dimensional Lie group acting smoothly on a manifold M. Without significant loss of generality, we assume that G acts locally effectively on subsets, [25]. Let $G^{(n)}$ denote the n^{th} order prolonged action of G on the jet bundle J^n. Following Cartan, [11, 12, 30], we denote the image of an n-jet $z^{(n)}$ under the prolonged group action by the corresponding capital letter $Z^{(n)} = g^{(n)} \cdot z^{(n)}$, $g^{(n)} \in G^{(n)}$. The *regular subset* $\mathcal{V}^n \subset \mathrm{J}^n$ is the open subset where $G^{(n)}$ acts locally freely and regularly. Thus the orbits of points in $\mathcal{V}^n$ under the prolonged action are of dimension $r = \dim G$. In [20] it is shown that if the action of G is locally effective on all open subsets of M, then $\mathcal{V}^n$ is nonempty and dense for n sufficiently large.

Definition 3.1. An n^{th} *order (right-equivariant) moving frame* is a map $\rho^{(n)} \colon \mathrm{J}^n \to G$ which is (locally) G-equivariant, i.e.,

$$\rho^{(n)}(g^{(n)} \cdot z^{(n)}) = \rho^{(n)}(z^{(n)}) \cdot g^{-1}, \qquad z^{(n)} \in \mathrm{J}^n, \qquad g \in G,$$

with respect to the prolonged action of $G^{(n)}$ on J^n, and the right multiplication action of G on itself. Given a sequence of moving frames $\rho^{(n)}$ consistent with the jet projections one obtains the *(infinite order) moving frame* $\rho = \rho^{(\infty)} \colon \mathrm{J}^\infty \to G$ as the projective limit.

The fundamental existence theorem for moving frames is as follows, [14].

Theorem 3.2. If G acts on M, then an n^{th} order moving frame exists in a neighborhood of $z^{(n)} \in \mathrm{J}^n$ if and only if $z^{(n)} \in \mathcal{V}^n$ is a regular jet.

In applications, the construction of a moving frame is based on Cartan's method of *normalization*, [10, 14], which requires the choice of a (local) cross-section $\mathcal{K}^n \subset \mathcal{V}^n$ to the group orbits. For expository purposes, we assume that $\mathcal{K}^n$ is a global cross-section, which may require shrinking the domain $\mathcal{V}^n \subset \mathrm{J}^n$ of regular jets.

Theorem 3.3. Let G act freely and regularly on $\mathcal{V}^n \subset \mathrm{J}^n$. Let $\mathcal{K}^n \subset \mathcal{V}^n$ be a cross-section to the group orbits. For $z^{(n)} \in \mathcal{V}^n$, let $g = \rho^{(n)}(z^{(n)})$ be the unique group element whose prolongation maps $z^{(n)}$ to the cross-section: $g^{(n)} \cdot z^{(n)} \in \mathcal{K}^n$. Then $\rho^{(n)} \colon \mathrm{J}^n \to G$ is a right equivariant moving frame for the group action.

The derivation of a moving frame involves three steps:

1. Compute the explicit local coordinate formulas for the prolonged group transformations
$$w^{(n)}(g, z^{(n)}) = Z^{(n)} = g^{(n)} \cdot z^{(n)}. \tag{3.1}$$
2. Choose (typically) a coordinate cross-section $\mathcal{K}^n = \{z_1 = c_1, \ldots, z_r = c_r\}$ obtained by setting $r = \dim G$ of the components of $z^{(n)} = (x, u^{(n)})$ equal to constants.
3. Using the labeling $w_1, \ldots, w_r$ for the components of the transformed cross-section, solve the *normalization equations*
$$w_1(g, z^{(n)}) = c_1 \qquad \cdots \qquad w_r(g, z^{(n)}) = c_r, \tag{3.2}$$
for the group parameters $g = (g_1, \ldots, g_r)$ in terms of the coordinates $z^{(n)}$.

Theorem 3.4. If $g = \rho^{(n)}(z^{(n)})$ is the moving frame solution to the normalization equations (3.2), then the components of

$$I^{(n)}(z^{(n)}) = w^{(n)}(\rho^{(n)}(z^{(n)}), z^{(n)})$$

form a complete system of differential invariants on the open subset of J^n where the moving frame is defined.

Note that the r invariants

$$I_1 = w_1(\rho^{(n)}(z^{(n)}), z^{(n)}) = c_1 \qquad \ldots \qquad I_r = w_r(\rho^{(n)}(z^{(n)}), z^{(n)}) = c_r \tag{3.3}$$

defining the cross-section (3.2) are constant. Those invariants are known as the *phantom invariants.*

Example 3.5. We consider the action of the Euclidean group $SE(2)$ on planar curves:

$$X = x\cos\phi - u\sin\phi + a, \qquad U = x\sin\phi + u\cos\phi + b, \qquad \phi, a, b \in \mathbb{R}. \tag{3.4}$$

The prolonged action

$$U_X = \frac{\sin\phi + u_x\cos\phi}{\cos\phi - u_x\sin\phi}, \qquad U_{XX} = \frac{u_{xx}}{(\cos\phi - u_x\sin\phi)^3}, \qquad U_{XXX} = \cdots,$$

is computed through implicit differentiation. A well known moving frame for this group action, [13, 14, 15, 19, 20, 27], follows from the cross-section normalization

$$X = 0, \qquad U = 0, \qquad U_X = 0.$$

Solving for the group parameters $g = (\phi, a, b)$ leads to the right-equivariant[2] moving frame

$$\phi = -\tan^{-1} u_x, \qquad a = -\frac{x + u u_x}{\sqrt{1 + u_x^2}}, \qquad b = \frac{x u_x - u}{\sqrt{1 + u_x^2}}. \tag{3.5}$$

The fundamental normalized differential invariants for the moving frame (3.5) are

$$X \mapsto H = 0, \qquad U \mapsto I_0 = 0, \qquad U_X \mapsto I_1 = 0,$$
$$U_{XX} \mapsto I_2 = \kappa = \frac{u_{xx}}{(1 + u_x^2)^{3/2}}, \quad U_{XXX} \mapsto I_3 = \kappa_s, \quad U_{XXXX} \mapsto I_4 = \kappa_{ss} + 3\kappa^3,$$

and so on. Here $\kappa_s = \mathcal{D}\kappa$ and $\kappa_{ss} = (\mathcal{D})^2 \kappa$ where $\mathcal{D} = (1 + u_x^2)^{-1/2} D_x$ is the Euclidean arc length derivative.

It is useful to adopt the viewpoint that a moving frame is a section of a certain bundle over J^n, called the lifted bundle.

Definition 3.6. The n^{th} *lifted bundle* consists of the bundle $\pi^n \colon \mathcal{B}^n = \mathrm{J}^n \times G \to \mathrm{J}^n$, with the *lifted prolonged group action*

$$g \cdot (z^{(n)}, h) = (g^{(n)} \cdot z^{(n)}, h \cdot g^{-1}), \qquad g \in G, \qquad (z^{(n)}, h) \in \mathcal{B}^n.$$

Taking the projective limit of the $\mathcal{B}^n$, we obtain the lifted bundle $\pi \colon \mathcal{B}^\infty = \mathrm{J}^\infty \times G \to \mathrm{J}^\infty$.

The components of the evaluation map (3.1) provide a complete system of lifted differential invariants on $\mathcal{B}^n$. In the projective limit, we write $w = w^{(\infty)} \colon \mathcal{B}^\infty \to \mathrm{J}^\infty$. This endows $\mathcal{B}^\infty$ with a groupoid structure, [21, 22, 30],

$$\begin{tikzcd} & \mathcal{B}^\infty \arrow[dl, "\pi"'] \arrow[dr, "w"] & \\ \mathrm{J}^\infty & & \mathrm{J}^\infty. \end{tikzcd}$$

An infinite order moving frame $\rho \colon \mathrm{J}^\infty \to G$ serves to define a local G-equivariant section $\sigma \colon \mathrm{J}^\infty \to \mathcal{B}^\infty$:

$$\sigma(z^{(\infty)}) = (z^{(\infty)}, \rho(z^{(\infty)})). \tag{3.6}$$

Let $\widehat{\boldsymbol{\Omega}}^*$ denote the space of differential forms on $\mathcal{B}^\infty$, which are called *lifted differential forms*. A coframe for $\widehat{\boldsymbol{\Omega}}^*$ consists of the horizontal and contact one-forms (2.1), (2.2), and the Maurer–Cartan forms $\mu^1, \ldots, \mu^r$ on G. To simplify notation, we identify a form on either J^∞ or G and its pull-back to $\mathcal{B}^\infty$ under the standard Cartesian projections. The Cartesian product structure $\mathcal{B}^\infty = \mathrm{J}^\infty \times G$ induces a bigrading on $\widehat{\boldsymbol{\Omega}}^* = \oplus_{k,l} \widehat{\boldsymbol{\Omega}}^{k,l}$, where $\widehat{\boldsymbol{\Omega}}^{k,l}$ denotes the space of forms which consist of combinations of wedge products of k jet components (either dx^i or θ^α_J) and l Maurer–Cartan forms μ^k. Let $\widehat{\boldsymbol{\Omega}}^*_J = \oplus_k \widehat{\boldsymbol{\Omega}}^{k,0}$ denote the space of pure jet forms on $\mathcal{B}^\infty$. A jet form may depend on group parameters, but does not contain Maurer–Cartan forms. Let $\pi_J \colon \widehat{\boldsymbol{\Omega}}^* \to \widehat{\boldsymbol{\Omega}}^*_J$ denote the *jet projection*, obtained by equating all Maurer–Cartan forms to zero.

[2]This moving frame is only locally equivariant, since there remains an ambiguity of π in the prescription of the rotation angle. We ignore this technical point here and refer to [26] for a detailed discussion.

4 The Invariant Variational Bicomplex

The theory of moving frames provides a process for *invariantizing* an arbitrary differential jet form. The bigrading of the variational bicomplex may be invariantized to produce a new bigrading and corresponding splitting of the differential, comprising the *invariant variational bicomplex* of Kogan and Olver, [19, 20]. For projectable group actions this new structure agrees with the old. For non-projectable actions, the new bigrading is different and the differential splits into three components, giving the *invariant variational bicomplex* the structure of a "quasi-tricomplex" and not a bicomplex proper. We remark that, although Kogan and Olver consider arbitrary differential forms, only the actually invariant forms in the invariant variational bicomplex are needed for the present considerations, so our definition of invariant variational bicomplex differs from that of [19, 20].

Definition 4.1. A locally defined differential form $\Omega \in \mathbf{\Omega}^*$ is said to be G-invariant if

$$(g^{(\infty)})^*\Omega = \Omega, \qquad \forall\, g \in G.$$

The collection of G-invariant differential forms is denoted by $\mathbf{\Omega}^*_G$.

Definition 4.2. The *invariantization* of a differential form Ω on J^∞ is the invariant differential form

$$\iota(\Omega) = \sigma^* \left(\pi_J(w^*\Omega)\right).$$

Lemma 4.3. The invariantization map ι defines a projection, $\iota^2 = \iota$, from the space of differential forms $\mathbf{\Omega}^*$ onto the space of invariant differential forms $\mathbf{\Omega}^*_G$.

In terms of the local coordinates $z^{(\infty)} = (x, u^{(\infty)})$, define the *invariant horizontal one-forms*

$$\varpi^i = \iota(dx^i), \qquad i = 1, \dots, p \tag{4.1}$$

and the fundamental *invariant contact forms*

$$\vartheta^\alpha_J = \iota(\theta^\alpha_J), \qquad \alpha = 1, \dots, q, \qquad \#J \geq 0. \tag{4.2}$$

It is important to note that if the group action is non-projectable, then the invariant horizontal one-forms (4.1) are not purely horizontal forms. If we decompose them into horizontal and contact components

$$\varpi^i = \omega^i + \eta^i, \qquad \text{where} \qquad \omega^i = \pi_H(\varpi^i), \quad \eta^i = \pi_V(\varpi^i), \tag{4.3}$$

their horizontal components $\omega^i \in \mathbf{\Omega}^{1,0}$ are the usual contact invariant horizontal forms, [14]. The invariant contact forms (4.2) are in all cases genuine contact forms and form a basis for the full contact ideal.

Example 4.4. Consider again the planar Euclidean group $SE(2)$ of Example 3.5. To obtain the invariant horizontal form (4.1), apply the invariantization map to dx:

$$\begin{aligned}
\iota(dx) &= \sigma^*(\pi_J(w^* dx)) \\
&= \sigma^*\big(\pi_J\big(\cos\phi\, dx - \sin\phi\, du - (x\sin\phi + u\cos\phi)d\phi + da\big)\big) \\
&= \sigma^*(\cos\phi\, dx - \sin\phi\, du) \\
&= \sigma^*\big((\cos\phi - u_x\sin\phi)dx - (\sin\phi)\theta\big),
\end{aligned}$$

where $\theta = du - u_x dx$ is the usual zero order basic contact form. Pulling back via the moving frame (3.5) leads to the invariant horizontal one-form

$$\varpi = \omega + \eta = \sqrt{1+u_x^2}\, dx + \frac{u_x}{\sqrt{1+u_x^2}}\, \theta, \tag{4.4}$$

which is a sum of the contact-invariant arc length form $\omega = ds = \sqrt{1+u_x^2}\, dx$ along with a contact correction term $\eta = u_x(1+u_x^2)^{-1/2}\theta$. The invariantization of the contact forms yields

$$\begin{gathered} \vartheta = \frac{\theta}{\sqrt{1+u_x^2}}, \qquad \vartheta_1 = \frac{(1+u_x^2)\theta_x - u_x u_{xx}\theta}{(1+u_x^2)^2}, \\ \vartheta_2 = \frac{(1+u_x^2)^2\theta_{xx} - 3u_x u_{xx}(1+u_x^2)\theta_x + (3u_x^2 u_{xx}^2 - u_x(1+u_x^2)u_{xxx})\theta}{(1+u_x^2)^{7/2}}, \end{gathered} \tag{4.5}$$

and so on.

Theorem 4.5. The invariant horizontal and contact one-forms (4.1), (4.2) form an invariant coframe on the domain of definition $\mathcal{V}^\infty \subset \mathrm{J}^\infty$ of the moving frame.

By virtue of Theorem 4.5, proved in [14], any one-form can be uniquely decomposed into a linear combination of invariant horizontal and invariant contact one-forms. These two components are called the *invariant horizontal* and *invariant vertical* components of the forms. In this manner, the invariant coframe (4.1), (4.2) is used to bigrade the space of differential forms on J^∞:

$$\mathbf{\Omega}^* = \bigoplus_{r,s} \widetilde{\mathbf{\Omega}}^{r,s},$$

where $\widetilde{\mathbf{\Omega}}^{r,s}$ is the space of forms of invariant horizontal degree r and invariant vertical degree s.

Let

$$\pi_{r,s} \colon \mathbf{\Omega} \to \mathbf{\Omega}^{r,s}, \qquad \widetilde{\pi}_{r,s} \colon \mathbf{\Omega} \to \widetilde{\mathbf{\Omega}}^{r,s} \tag{4.6}$$

denote, respectively, projection of arbitrary differential forms onto the ordinary and the invariant (r,s)-bigrade. Because of (4.3), horizontal and invariant horizontal forms differ only by contact forms, so the restrictions of the projections (4.6)

$$\pi_{r,s} \colon \widetilde{\mathbf{\Omega}}^{r,s} \to \mathbf{\Omega}^{r,s}, \qquad \widetilde{\pi}_{r,s} \colon \mathbf{\Omega}^{r,s} \to \widetilde{\mathbf{\Omega}}^{r,s} \tag{4.7}$$

are mutually inverse.

Invariantization defines a map

$$\iota \colon \mathbf{\Omega}^{r,s} \to \widetilde{\mathbf{\Omega}}^{r,s}_G \subset \widetilde{\mathbf{\Omega}}^{r,s}$$

that takes an ordinary form of bigrade (r,s) and produces an invariant form of invariant bigrade (r,s). In general this map does not commute with the exterior derivative:

$$d\iota(\Omega) \neq \iota(d\Omega).$$

Computation of the correction terms for this lack of commutativity is central to the construction of the invariant variational bicomplex.

Before discussing these correction terms, we briefly recall notation for infinitesimal generators and their prolongations. A Lie algebra element $\mathbf{v} \in \mathfrak{g}$ generates a vector field $\widehat{\mathbf{v}}$ (an *infinitesimal generator*) on M through the usual process:

$$\widehat{\mathbf{v}} = \frac{d}{d\epsilon}\bigg|_{\epsilon=0} (\exp(\epsilon \mathbf{v}) \cdot z). \tag{4.8}$$

Due to the local effectiveness of the action of G, the Lie algebra $\mathfrak{g}$ may be identified with the Lie algebra of infinitesimal generators on M. Thus we drop the notational distinction between $\mathbf{v}$ and $\widehat{\mathbf{v}}$. Given a basis $\mathbf{v}_1, \ldots, \mathbf{v}_r$ for $\mathfrak{g}$ there is a corresponding Lie algebra of infinitesimal generators on M with generators

$$\mathbf{v}_\kappa = \sum_{i=1}^{p} \xi_{\kappa,i}(x,u)\frac{\partial}{\partial x^i} + \sum_{\alpha=1}^{q} \phi_{\kappa,\alpha}(x,u)\frac{\partial}{\partial u^\alpha}, \qquad \kappa = 1, \ldots, r. \tag{4.9}$$

The expressions for the infinitesimal generators of the prolonged group action $G^{(n)}$

$$\mathbf{v}_\kappa^{(n)} = \mathbf{v}_\kappa + \sum_{\alpha=1}^{q} \sum_{\#J \geq 1}^{n} \phi_{\kappa,\alpha}^J(x, u^{(n)})\frac{\partial}{\partial u_J^\alpha}, \qquad \kappa = 1, \ldots, r,$$

are given by the standard recursive formula, [23],

$$\phi_{\kappa,\alpha}^{J,j} = D_j \phi_{\kappa,\alpha}^J - \sum_{i=1}^{p} D_j \xi_{\kappa,i} \cdot u_{J,i}^\alpha.$$

The infinite prolongation $\mathbf{v}^{(\infty)}$ may be found in a similar fashion.

The following lemma, called the *recurrence formula*, exhibits the correction terms we seek. A proof may be found in [20].

Lemma 4.6. Let $\mu^1, \ldots, \mu^r \in \mathfrak{g}^*$ be the Maurer–Cartan forms dual to $\mathbf{v}_1, \ldots, \mathbf{v}_r \in \mathfrak{g}$. If Ω is any differential form on J^∞,

$$d\iota(\Omega) = \iota(d\Omega) + \sum_{\kappa=1}^{r} \nu^\kappa \wedge \iota[\mathcal{L}_{\mathbf{v}_\kappa^{(\infty)}}(\Omega)] \tag{4.10}$$

where $\nu^\kappa = \rho^*(\mu^\kappa)$ are the pull-backs of the Maurer–Cartan forms μ^κ via the moving frame $\rho\colon J^\infty \to G$ and $\mathcal{L}_{\mathbf{v}_\kappa^{(\infty)}}(\Omega)$ is the Lie derivative of Ω with respect to $\mathbf{v}_\kappa^{(\infty)}$.

Remark 4.7. An important observation is that the differential forms $\nu^1, \ldots, \nu^r$ can be determined directly from the recurrence formula (4.10). Indeed, for the r phantom invariants (3.3), the left-hand side of (4.10) is identically zero, and those r equations can be used to solve for the r unknown differential forms ν^κ. The solution to the system of equations is guaranteed by our regularity assumptions on the group action.

With the observation that for $\Omega \in \mathbf{\Omega}^{r,s}$, $d\Omega \in \mathbf{\Omega}^{r+1,s} \oplus \mathbf{\Omega}^{r,s+1}$ and $\mathbf{v}^\kappa(\Omega) \in \mathbf{\Omega}^{r,s} \oplus \mathbf{\Omega}^{r-1,s+1}$ it follows from (4.10) that

$$d\iota(\Omega) \in \widetilde{\mathbf{\Omega}}_G^{r+1,s} \oplus \widetilde{\mathbf{\Omega}}_G^{r,s+1} \oplus \widetilde{\mathbf{\Omega}}_G^{r-1,s+2} \subset \widetilde{\mathbf{\Omega}}^{r+1,s} \oplus \widetilde{\mathbf{\Omega}}^{r,s+1} \oplus \widetilde{\mathbf{\Omega}}^{r-1,s+2},$$

with the convention that $\widetilde{\mathbf{\Omega}}^{-1,s} = 0$, $s \geq 0$. In fact, since any (possibly non-invariant) $\Omega \in \widetilde{\mathbf{\Omega}}^{r,s}$ is a linear combination with function coefficients of invariant forms of invariant bigrade (r, s), $d\Omega$ decomposes similarly:

$$d\Omega = d_{\mathcal{H}}\Omega + d_{\mathcal{V}}\Omega + d_{\mathcal{W}}\Omega,$$
$$d_{\mathcal{H}}\Omega \in \widetilde{\mathbf{\Omega}}^{r+1,s}, \quad d_{\mathcal{V}}\Omega \in \widetilde{\mathbf{\Omega}}^{r,s+1}, \quad d_{\mathcal{W}}\Omega \in \widetilde{\mathbf{\Omega}}^{r-1,s+2}.$$

This gives the invariant bigraded forms the structure of a *quasi-tricomplex*:

$$\begin{gathered} d_{\mathcal{H}}^2 = 0, \quad d_{\mathcal{W}}^2 = 0, \\ d_{\mathcal{H}}d_{\mathcal{V}} + d_{\mathcal{V}}d_{\mathcal{H}} = 0, \quad d_{\mathcal{V}}d_{\mathcal{W}} + d_{\mathcal{W}}d_{\mathcal{V}} = 0, \quad d_{\mathcal{V}}^2 + d_{\mathcal{H}}d_{\mathcal{W}} + d_{\mathcal{W}}d_{\mathcal{H}} = 0. \end{gathered} \tag{4.11}$$

If the action is projectable, Lie differentiation by infinitesimal generators will preserve the ordinary bigrading, resulting in $d_{\mathcal{W}} = 0$ and reducing the above "quasi-tricomplex" structure to an ordinary bicomplex (2.4) in $d_{\mathcal{H}}$ and $d_{\mathcal{V}}$.

We now introduce the invariant variational bicomplex and the invariant Euler–Lagrange complex. For $s \geq 1$, define the spaces of G-invariant source forms and the quotient projections

$$\widetilde{\mathcal{F}}_G^s = \widetilde{\mathbf{\Omega}}_G^{p,s} / d_{\mathcal{H}}(\widetilde{\mathbf{\Omega}}_G^{p-1,s}) \qquad \text{and} \qquad \widetilde{\pi} \colon \widetilde{\mathbf{\Omega}}_G^{p,s} \to \widetilde{\mathcal{F}}_G^s. \tag{4.12}$$

Let $\widetilde{E} = \widetilde{\pi} \circ d_{\mathcal{V}} \colon \widetilde{\mathbf{\Omega}}_G^{p,0} \to \widetilde{\mathcal{F}}_G^1$ and define $\delta_{\mathcal{V}} = \widetilde{\pi} \circ d_{\mathcal{V}} \colon \widetilde{\mathcal{F}}_G^s \to \widetilde{\mathcal{F}}_G^{s+1}$ where the latter map is understood to act on equivalence class representatives. As in the ordinary case, this action is well defined by the anticommutativity of $d_{\mathcal{H}}$ and $d_{\mathcal{V}}$. That $\delta_{\mathcal{V}}$ is a boundary operator follows from the implication of the relations (4.11), as $d_{\mathcal{V}}^2 \widetilde{\Omega} = -d_{\mathcal{H}}d_{\mathcal{W}}\widetilde{\Omega}$ for $\widetilde{\Omega}$ of maximum invariant horizontal degree.

Definition 4.8. The (augmented) *invariant variational bicomplex* is the quasi-tricomplex

$$(\widetilde{\mathbf{\Omega}}_G^{*,*}, \{d_{\mathcal{H}}, d_{\mathcal{V}}, d_{\mathcal{W}}\}).$$

to which we add the vertical complex $(\widetilde{\mathcal{F}}_G^*, \delta_{\mathcal{V}})$ as in Definition 2.2.

Remark 4.9. As mentioned earlier, our definition of the invariant variational bicomplex differs from the original definition of Kogan and Olver, [19, 20], in that we consider only invariant forms.

Following the example of the ordinary variational bicomplex, an edge complex, called the *invariant Euler–Lagrange complex*, may be constructed for the invariant variational bicomplex.

Definition 4.10. The *invariant Euler–Lagrange complex* is the edge complex

$$0 \longrightarrow \mathbb{R} \longrightarrow \widetilde{\mathbf{\Omega}}_G^{0,0} \xrightarrow{d_{\mathcal{H}}} \widetilde{\mathbf{\Omega}}_G^{1,0} \xrightarrow{d_{\mathcal{H}}} \cdots \xrightarrow{d_{\mathcal{H}}} \widetilde{\mathbf{\Omega}}_G^{p,0} \xrightarrow{\tilde{E}} \widetilde{\mathcal{F}}_G^1 \xrightarrow{\delta_{\mathcal{V}}} \widetilde{\mathcal{F}}_G^2 \xrightarrow{\delta_{\mathcal{V}}} \cdots.$$

Using the equivariant moving frame method, the explicit expression for the *invariant Euler–Lagrange operator* $\widetilde{E} \colon \widetilde{\mathbf{\Omega}}_G^{p,0} \to \widetilde{\mathcal{F}}_G^1$, was discovered by Kogan and Olver, [20].

5 Local Exactness of the Interior Rows of the Invariant Variational Bicomplex

In this section the local exactness of the interior rows of the invariant variational bicomplex is established. Following [5], an invariant connection is introduced and used to construct invariant homotopy operators for these rows. To define the invariant connection we first introduce a G-invariant splitting of the tangent bundle $T\mathrm{J}^\infty$ dual to the invariant bigrading on $\boldsymbol{\Omega}^*$.

First, recall that a total vector field on J^∞ is one which is annihilated by any contact form. The space of total vector fields forms a subbundle $\mathbf{H}$ of $T\mathrm{J}^\infty$. In the local coordinate system $M \simeq X \times U$, the total differential operators $D_1, \ldots, D_p$ in (2.5) form a basis of total vector fields. When a moving frame exists, we can replace the standard basis of total vector fields by the invariant total differential operators $\mathcal{D}_1, \ldots, \mathcal{D}_p$ dual to the invariant horizontal forms $\varpi^1, \ldots, \varpi^p$ and

$$\mathbf{H} = \mathrm{span}\{D_1, \ldots, D_p\} = \mathrm{span}\{\mathcal{D}_1, \ldots, \mathcal{D}_p\}.$$

Now, let $\mathbf{V}_G$ be the subbundle of *G-invariant vertical vector fields* defined as the span of the vector fields V_α^J dual to the basic invariant contact forms ϑ_J^α. Denoting the standard pairing between $T\mathrm{J}^\infty$ and $\boldsymbol{\Omega}^*$ by $\langle\cdot;\cdot\rangle$, the invariant vertical vector fields V_α^J are defined by the relations

$$\langle V_\alpha^J; \varpi^i \rangle = 0, \qquad \langle V_\alpha^J; \vartheta_K^\beta \rangle = \delta_\alpha^\beta \delta_K^J,$$

where δ_α^β, δ_K^J are Kronecker symbols. Let $\mathbf{V}$ denote the subbundle of (vertical) vector fields annihilated by $d\pi_X^\infty \colon T\mathrm{J}^\infty \to TX$. When the group action is not projectable

$$\mathbf{V}_G \neq \mathbf{V}.$$

Example 5.1. The vector fields dual to the invariant coframe (4.4), (4.5) are given by the arc length derivative

$$\mathcal{D} = \frac{D_x}{\sqrt{1+u_x^2}} \tag{5.1}$$

and the invariant vertical vector fields

$$\begin{gathered} V^0 = \frac{1}{\sqrt{1+u_x^2}}\left(-u_x\frac{\partial}{\partial x} + \frac{\partial}{\partial u}\right), \qquad V^1 = (1+u_x^2)\frac{\partial}{\partial u_x} + 3u_x u_{xx}\frac{\partial}{\partial u_{xx}} + \cdots, \\ V^2 = (1+u_x^2)^{3/2}\frac{\partial}{\partial u_{xx}} + \cdots, \qquad \ldots . \end{gathered} \tag{5.2}$$

We note that $d\pi_X(V^k) = 0$, for $k \geq 1$. In general, this equality always holds, that is $d\pi_X(V_\alpha^J) = 0$ whenever $\#J \geq 1$. For a non-projectable action, the only invariant vertical vector fields with non trivial horizontal component are the vectors V_α dual to the zero order contact forms ϑ^α.

Given the subbundles $\mathbf{H}$ and $\mathbf{V}_G$, the tangent bundle $T\mathrm{J}^\infty$ decomposes into a G-invariant direct sum

$$T\mathrm{J}^\infty = \mathbf{H} \oplus \mathbf{V}_G,$$

and we can define the projections

$$\mathrm{Tot}\colon \mathcal{X}(\mathrm{J}^\infty) \to \Gamma(\mathrm{J}^\infty, \mathbf{H}) \qquad \text{and} \qquad \mathrm{Vert}_G\colon \mathcal{X}(\mathrm{J}^\infty) \to \Gamma(\mathrm{J}^\infty, \mathbf{V}_G)$$

of a vector field onto its G-invariant horizontal and vertical components. Recall the notation $\mathcal{X}$ for the collection of vector fields on J^∞ and Γ for sections of $\mathbf{H}$ or $\mathbf{V}_G$ over J^∞.

Definition 5.2. A *horizontal connection* on the bundle $\mathbf{H}$ of total vector fields is an $\mathbb{R}$-bilinear map which assigns to a pair of total vector fields X and Y a total vector field $\widehat{\nabla}_X Y$ satisfying

a) $\widehat{\nabla}_{fX} Y = f\widehat{\nabla}_X Y$,

b) $\widehat{\nabla}_X(fY) = X(f)Y + f\widehat{\nabla}_X Y$,

where f is any smooth differential function.

Definition 5.3. The connection $\widehat{\nabla}$ is said to be *torsion-free* if

$$\widehat{\nabla}_X Y - \widehat{\nabla}_Y X = [X, Y].$$

Definition 5.4. The connection $\widehat{\nabla}$ is G-invariant if

$$\mathcal{L}_{\mathbf{v}^{(\infty)}}(\widehat{\nabla}_X Y) = \widehat{\nabla}_{(\mathcal{L}_{\mathbf{v}^{(\infty)}} X)} Y + \widehat{\nabla}_X(\mathcal{L}_{\mathbf{v}^{(\infty)}} Y) \tag{5.3}$$

for all infinitesimal generators $\mathbf{v} \in \mathfrak{g}$ and total vector fields $X, Y \in \mathbf{H}$. Note that the right-hand side of (5.3) is well-defined since $\mathcal{L}_{\mathbf{v}^{(\infty)}} X$ and $\mathcal{L}_{\mathbf{v}^{(\infty)}} Y$ are total vector fields.

Remark 5.5. Invariant, torsion-free horizontal connections on $\mathbf{H}$ can be constructed for any group action admitting p functionally independent differential invariants $I^i(x, u^{(\infty)})$, $i = 1, \ldots, p$. Let $\{\mathcal{R}_1, \ldots, \mathcal{R}_p\}$ be the basis for the distribution of total vector fields dual to the basis of invariant horizontal forms $\{d_\mathcal{H} I^1, \ldots d_\mathcal{H} I^p\}$. As the forms $d_\mathcal{H} I^i$ are $d_\mathcal{H}$-closed and G-invariant, the vector fields $\mathcal{R}_i$ commute among themselves and with the infinitesimal symmetry generators, that is

$$[\mathcal{R}_i, \mathcal{R}_j] = 0 \qquad \text{and} \qquad [\mathbf{v}^{(\infty)}, \mathcal{R}_i] = 0$$

for all i, j and $\mathbf{v} \in \mathfrak{g}$. Define $\widehat{\nabla}$ to be the unique horizontal connection on horizontal vector fields satisfying

$$\widehat{\nabla}_{\mathcal{R}_i} \mathcal{R}_j = 0, \qquad \text{for all} \qquad 1 \le i, j \le p.$$

Then this connection is torsion-free and G-invariant.

We extend the connection $\widehat{\nabla}$ to the full tangent bundle of $T\mathrm{J}^\infty$, in a G-invariant manner, by setting

$$\nabla_X Z = \widehat{\nabla}_X \mathrm{Tot}\, Z + \mathrm{Vert}_G\,[X, \mathrm{Vert}_G\, Z]. \tag{5.4}$$

To simplify the notation, let

$$\nabla_i = \nabla_{\mathcal{R}_i}.$$

The next lemma is a direct consequence of the G-invariance of ∇.

Lemma 5.6. Let $\widetilde{\Omega} \in \widetilde{\mathbf{\Omega}}^{r,s}_G$ be an invariant differential form. Then for all $\mathbf{v} \in \mathfrak{g}$

$$\mathcal{L}_{\mathbf{v}^{(\infty)}}(\nabla_i \widetilde{\Omega}) = 0, \qquad i = 1, \dots, p, \tag{5.5}$$

that is, $\nabla_i \widetilde{\Omega} \in \widetilde{\mathbf{\Omega}}^{r,s}_G$ is an invariant differential form.

The invariant connection may be used to conveniently write the invariant horizontal differential of a form, [5].

Lemma 5.7. Let $\Omega \in \mathbf{\Omega}^k$ and ∇ be an invariant connection constructed as above. Then the invariant horizontal differential of Ω is given by

$$d_{\mathcal{H}}\Omega = \sum_{i=1}^{p} d_{\mathcal{H}} I^i \wedge \nabla_i(\Omega). \tag{5.6}$$

The horizontal and invariant horizontal differentials are related to each other through the projection maps (4.7), [20].

Lemma 5.8. The horizontal and invariant horizontal differentials satisfy the relations

$$\pi_{r+1,s} \circ d_{\mathcal{H}} = d_H \circ \pi_{r,s}, \qquad \widetilde{\pi}_{r+1,s} \circ d_H = d_{\mathcal{H}} \circ \widetilde{\pi}_{r,s},$$

for any $0 \le r \le p$ and $s \ge 0$.

Example 5.9. For the Euclidean group action $SE(2)$, the invariant connection (5.4) is constructed as follows. Since the arc length derivative (5.1) commutes with itself there is no need to introduce a new commuting basis of invariant total derivatives. Using the recurrence relation (4.6) we deduce the structure equations

$$\begin{aligned}
d\varpi &= \kappa\varpi \wedge \vartheta + \vartheta_1 \wedge \vartheta, \\
d\vartheta &= \varpi \wedge \vartheta_1, \\
d\vartheta_1 &= \varpi \wedge \vartheta_2 - \kappa^2\varpi \wedge \vartheta - \kappa\vartheta_1 \wedge \vartheta, \\
d\vartheta_2 &= \varpi \wedge \vartheta_3 - 3\kappa^2\varpi \wedge \vartheta_1 - \kappa\kappa_s\varpi \wedge \vartheta - \kappa_s\vartheta_1 \wedge \vartheta, \\
&\ \ \vdots
\end{aligned} \tag{5.7}$$

By duality, we obtain the commutator relations

$$\begin{gathered}
[\mathcal{D}, \mathcal{D}] = 0, \qquad [\mathcal{D}, V^0] = -\kappa\mathcal{D} + \kappa^2 V^1 + \kappa\kappa_s V^2 + \cdots, \\
[\mathcal{D}, V^1] = -V^0 + 3\kappa^2 V^2 + \cdots, \qquad [\mathcal{D}, V^2] = -V^1 + \cdots, \qquad \dots.
\end{gathered} \tag{5.8}$$

among the invariant vector fields (5.1), (5.2). Substituting (5.8) into the definition of the invariant connection (5.4) we obtain

$$\begin{gathered}
\nabla_{\mathcal{D}}\mathcal{D} = 0, \qquad \nabla_{\mathcal{D}} V^0 = \kappa^2 V^1 + \kappa\kappa_s V^2 + \dots, \\
\nabla_{\mathcal{D}} V^1 = -V^0 + 3\kappa^2 V^2 + \cdots, \qquad \nabla_{\mathcal{D}} V^2 = -V^1 + \cdots,
\end{gathered} \tag{5.9}$$

from which we deduce the nonzero Christoffel symbols

$$\Gamma^1_0 = \kappa^2, \quad \Gamma^2_0 = \kappa\kappa_s, \quad \dots, \qquad \Gamma^0_1 = -1, \quad \Gamma^2_1 = 3\kappa^2, \quad \dots, \qquad \Gamma^1_2 = -1, \quad \dots, \tag{5.10}$$

where the subscript indexing $\mathcal{D}$ in Γ^k_{ij} is omitted. We now verify formula (5.6), with the slight difference that we have used ϖ and $\mathcal{D}$ rather than $d_\mathcal{H} I^i$ and $\mathcal{R}_i$ in the construction of the connection. Using the Christoffel symbols (5.10) we obtain

$$\begin{aligned} d_\mathcal{H}\varpi &= \varpi \wedge \nabla_\mathcal{D}\varpi = 0, \\ d_\mathcal{H}\vartheta &= \varpi \wedge \nabla_\mathcal{D}\vartheta = \varpi \wedge \vartheta_1, \\ d_\mathcal{H}\vartheta_1 &= \varpi \wedge \nabla_\mathcal{D}\vartheta_1 = \varpi \wedge (-\kappa^2\vartheta + \vartheta_2), \\ &\vdots \end{aligned}$$

which is, as expected, equal to the horizontal component of the structure equations (5.7).

Since for any contact one-form ϑ the equality

$$d_\mathcal{H}\vartheta = \sum_{i=1}^{p} d_\mathcal{H} I^i \wedge \mathcal{R}_i(\vartheta)$$

holds, [20], it follows from (5.6) that

$$\mathcal{R}_i(\vartheta) = \nabla_i(\vartheta). \tag{5.11}$$

We emphasize that (5.11) does not generally hold for horizontal one-forms. For example, for the Euclidean group $SE(2)$ acting on $\mathbb{R}^2$ we have

$$0 = \nabla_\mathcal{D}\varpi \neq \mathcal{D}(\varpi) = \kappa\vartheta,$$

where ϑ is the zero order invariant contact one-form given in (4.5). We now prove the main result of this section.

Theorem 5.10. Let G be a Lie group acting effectively on subsets of a manifold M. Then for each $s \geq 1$, the augmented horizontal complex

$$0 \longrightarrow \widetilde{\boldsymbol{\Omega}}^{0,s}_G \xrightarrow{d_\mathcal{H}} \widetilde{\boldsymbol{\Omega}}^{1,s}_G \xrightarrow{d_\mathcal{H}} \cdots \xrightarrow{d_\mathcal{H}} \widetilde{\boldsymbol{\Omega}}^{p,s}_G \xrightarrow{\widetilde{\pi}} \widetilde{\mathcal{F}}^s_G \longrightarrow 0 \tag{5.12}$$

is locally exact.

Proof. The regularity assumption on the action of G guarantees the existence of a moving frame, which is used to obtain p pairwise commuting invariant total differential operators $\mathcal{R}_1, \ldots, \mathcal{R}_p$ and define the invariant horizontal connection ∇ in (5.4). Using the invariant horizontal connection we now construct the *invariant interior Euler operator* $\widetilde{I}_\nabla \colon \widetilde{\boldsymbol{\Omega}}^{p,s}_G \to \widetilde{\boldsymbol{\Omega}}^{p,s}_G$. Let $\overline{V}^\alpha_J$ denote the invariant vertical vector fields dual to the basis of invariant contact one-forms $\mathcal{R}_J(\vartheta^\alpha)$. By virtue of (5.11), given $\widetilde{\Omega} \in \widetilde{\boldsymbol{\Omega}}^{p,s}$ we can write

$$\begin{aligned} \widetilde{\Omega} &= \frac{1}{s}\sum_J \sum_{\alpha=1}^{q} \mathcal{R}_J(\vartheta^\alpha) \wedge \left(\overline{V}^\alpha_J \lrcorner\, \widetilde{\Omega}\right) = \frac{1}{s}\sum_J \sum_{\alpha=1}^{q} \nabla_J(\vartheta^\alpha) \wedge \left(\overline{V}^\alpha_J \lrcorner\, \widetilde{\Omega}\right) \\ &= \frac{1}{s}\sum_J \sum_{\alpha=1}^{q} \nabla_J(\vartheta^\alpha \wedge \widetilde{F}^J_{\nabla,\alpha}(\widetilde{\Omega})), \end{aligned} \tag{5.13}$$

where the $\widetilde{F}^J_{\nabla,\alpha}$ are the interior Euler operators (2.9) expressed in terms of the connection $\nabla_1,\ldots,\nabla_p$; symbolically this is achieved by replacing the total differential operators D_J with ∇_J and the vector fields $\partial/\partial u^\alpha_J$ by $\overline{V}^\alpha_J$. Using (5.6) we can write

$$\widetilde{\Omega} = \widetilde{I}_\nabla(\widetilde{\Omega}) + d_{\mathcal{H}}(\widetilde{h}^{p,s}_\nabla(\widetilde{\Omega})), \tag{5.14}$$

where

$$\begin{aligned}\widetilde{I}_\nabla(\widetilde{\Omega}) =& \frac{1}{s}\sum_{\alpha=1}^{q}\vartheta^\alpha\wedge\widetilde{F}_{\nabla,\alpha}(\widetilde{\Omega}),\\ \widetilde{h}^{p,s}_\nabla(\widetilde{\Omega}) =& \frac{1}{s}\sum_J\sum_{j=1}^{p}\sum_{\alpha=1}^{q}\nabla_J\{\mathcal{R}_j\lrcorner[\vartheta^\alpha\wedge\widetilde{F}^{J,j}_{\nabla,\alpha}(\widetilde{\Omega})]\}.\end{aligned} \tag{5.15}$$

Now, let $\Omega\in\mathbf{\Omega}^{p,s}$, $\widetilde{\Omega}=\widetilde{\pi}_{p,s}(\Omega)$ and $I\colon\mathbf{\Omega}^{p,s}\to\mathbf{\Omega}^{p,s}$ be the standard (non-invariant) interior Euler operator defined by

$$I(\Omega) = \frac{1}{s}\sum_{\alpha=1}^{q}\theta^\alpha\wedge\Bigg[\sum_J(-D)_J\bigg(\frac{\partial}{\partial u^\alpha_J}\lrcorner\,\Omega\bigg)\Bigg]. \tag{5.16}$$

Then there exist differential forms $\eta\in\mathbf{\Omega}^{p-1,s}$ and $\widetilde{\eta}\in\widetilde{\mathbf{\Omega}}^{p,s}$ such that

$$\begin{aligned}I(\Omega)+d_H(\eta) = \Omega = \pi_{p,s}(\widetilde{\Omega}) &= \pi_{p,s}[\widetilde{I}_\nabla(\widetilde{\Omega})+d_{\mathcal{H}}(\widetilde{\eta})]\\ &= \pi_{p,s}\circ\widetilde{I}_\nabla\circ\widetilde{\pi}_{p,s}(\Omega)+d_H(\pi_{p-1,s}(\widetilde{\eta})).\end{aligned}$$

As $\pi_{p,s}\circ\widetilde{I}_\nabla\circ\widetilde{\pi}_{p,s}$ defines an interior Euler operator on $\mathbf{\Omega}^{p,s}$ it follows from [1, Proposition 5.55] that

$$I = \pi_{p,s}\circ\widetilde{I}_\nabla\circ\widetilde{\pi}_{p,s}. \tag{5.17}$$

The equality (5.17) implies that $\widetilde{I}_\nabla$ is independent of the connection and we write $\widetilde{I}_\nabla=\widetilde{I}$. Since $\ker I = d_H\mathbf{\Omega}^{p-1,s}$ we conclude from Lemma 5.8 that $\ker\widetilde{I}=d_{\mathcal{H}}\widetilde{\mathbf{\Omega}}^{p-1,s}$ and

$$\widetilde{I}(\widetilde{\mathbf{\Omega}}^{p,s}_G)\simeq\widetilde{\mathcal{F}}^s_G.$$

This shows that the invariant horizontal subcomplex

$$\widetilde{\mathbf{\Omega}}^{p-1,s}_G\xrightarrow{d_{\mathcal{H}}}\widetilde{\mathbf{\Omega}}^{p,s}_G\xrightarrow{\widetilde{\pi}}\widetilde{\mathcal{F}}^s_G\longrightarrow 0\,,\qquad s\geq 1,$$

is exact. For the first part of the invariant horizontal complex

$$0\longrightarrow\widetilde{\mathbf{\Omega}}^{0,s}_G\xrightarrow{d_{\mathcal{H}}}\widetilde{\mathbf{\Omega}}^{1,s}_G\xrightarrow{d_{\mathcal{H}}}\cdots\xrightarrow{d_{\mathcal{H}}}\widetilde{\mathbf{\Omega}}^{p,s}_G\,,\qquad s\geq 1,$$

invariant horizontal homotopy operators

$$\widetilde{h}^{r,s}_\nabla\colon\widetilde{\mathbf{\Omega}}^{r,s}_G\to\widetilde{\mathbf{\Omega}}^{r-1,s}_G,\qquad 1\leq r\leq p-1,$$

are similarly constructed:

$$\widetilde{h}^{r,s}_\nabla(\widetilde{\Omega}) = \frac{1}{s}\sum_{\#J=0}^{k-1}\sum_{j=1}^{p}\sum_{\alpha=1}^{q}\frac{\#J+1}{p-r+\#J+1}\nabla_J[\vartheta^\alpha\wedge\widetilde{F}^{J,j}_{\nabla,\alpha}(\widetilde{\Omega}_j)], \tag{5.18}$$

where $\widetilde{\Omega}_j=\mathcal{R}_j\lrcorner\,\widetilde{\Omega}$. $\square$

Example 5.11. Continuing with our running example, we illustrate the constructions of the last proof by first computing (5.14) explicitly for the invariant three-form $\widetilde{\Omega} = \vartheta_2 \wedge \vartheta_1 \wedge \varpi$. Beginning with the invariant interior Euler operator we have

$$\widetilde{I}_\nabla(\widetilde{\Omega}) = \frac{1}{2}\vartheta \wedge [V^0 \lrcorner\, \widetilde{\Omega} - \nabla_{\mathcal{D}}(V^1 \lrcorner\, \widetilde{\Omega}) - \nabla^2_{\mathcal{D}}(V^2 \lrcorner\, \widetilde{\Omega})] = \vartheta \wedge [2(\mathcal{D}^3\vartheta) + \kappa^2\vartheta_1] \wedge \varpi.$$

Next,

$$\widetilde{h}^{1,2}_\nabla(\widetilde{\Omega}) = \frac{1}{2}\left\{\mathcal{D} \lrcorner\, \vartheta \wedge \widetilde{F}^1_\nabla(\widetilde{\Omega}) + \nabla_{\mathcal{D}}[\mathcal{D} \lrcorner\, \vartheta \wedge \widetilde{F}^2_\nabla(\widetilde{\Omega})]\right\} = -\vartheta \wedge \mathcal{D}^2\vartheta,$$

and the equality

$$\begin{aligned}\widetilde{\Omega} &= \widetilde{I}_\nabla(\widetilde{\Omega}) + d_{\mathcal{H}}(\widetilde{h}^{1,2}_\nabla(\widetilde{\Omega})) = \vartheta \wedge [2(\mathcal{D}^3\vartheta) + \kappa^2\vartheta_1] \wedge \varpi + \varpi \wedge \nabla_{\mathcal{D}}(-\vartheta \wedge \mathcal{D}^2\vartheta) \\ &= \kappa^2\vartheta \wedge \vartheta_1 \wedge \varpi + (\mathcal{D}^2\vartheta) \wedge \vartheta_1 \wedge \varpi = \vartheta_2 \wedge \vartheta_1 \wedge \varpi\end{aligned}$$

is verified. We also check that (5.18) are homotopy operators for $\widetilde{\Omega} = \kappa\vartheta$, for example. Since $\widetilde{\Omega}$ is a contact form $\widetilde{h}^{0,1}_\nabla(\widetilde{\Omega}) = 0$. Thus

$$\begin{aligned}\widetilde{h}^{1,1}_\nabla(d_{\mathcal{H}}\widetilde{\Omega}) + d_{\mathcal{H}}(\widetilde{h}^{0,1}_\nabla(\widetilde{\Omega})) &= \widetilde{h}^{1,1}_\nabla(d_{\mathcal{H}}\widetilde{\Omega}) \\ &= \vartheta \wedge \widetilde{F}^1_\nabla(\kappa_s\vartheta + \kappa\vartheta_1) + \nabla_{\mathcal{D}}[\vartheta \wedge \widetilde{F}^2_\nabla(\kappa_s\vartheta + \kappa\vartheta_1) = \kappa\vartheta = \widetilde{\Omega}.\end{aligned}$$

6 The Local Cohomology of the Invariant Euler–Lagrange Complex

The purpose of this section is to establish an isomorphism between the invariant de Rham cohomology of J^∞ and the local cohomology of the invariant Euler–Lagrange complex. This isomorphism will be used in conjunction with the results of Section 7 to produce explicit examples of cohomology classes in the invariant Euler–Lagrange complex. Section 8 will be devoted to these examples.

Although the "snaking" arguments to follow are somewhat standard in appearance we include some details due to the appearance of the anomalous $d_{\mathcal{W}}$ operator. Recall the projections $\widetilde{\pi}^{r,s}$ and $\widetilde{\pi}$ from (4.7) and (4.12).

Lemma 6.1. Let $\gamma \in \mathbf{\Omega}^r_G$ be d-closed. If $r \leq p$ and $\widetilde{\pi}_{r,0}(\gamma) = 0$ or if $r = p + s$ and $(\widetilde{\pi} \circ \widetilde{\pi}_{p,s})(\gamma) = 0$, then γ is d-exact.

Proof. For $r \leq p$, write $\gamma = \gamma_1 + \gamma_2 + \cdots + \gamma_r$ where $\gamma_i \in \widetilde{\mathbf{\Omega}}^{r-i,i}_G$. Since γ is d-closed these forms satisfy

$$\begin{aligned}&d_{\mathcal{H}}\gamma_1 = 0, \\ &d_{\mathcal{H}}\gamma_2 + d_{\mathcal{V}}\gamma_1 = 0, \\ &d_{\mathcal{H}}\gamma_i + d_{\mathcal{V}}\gamma_{i-1} + d_{\mathcal{W}}\gamma_{i-2} = 0, \qquad i = 3, \ldots, r, \\ &d_{\mathcal{V}}\gamma_r + d_{\mathcal{W}}\gamma_{r-1} = 0.\end{aligned} \tag{6.1}$$

The exactness of the interior rows (5.12), combined with the equations (6.1) implies that there exist invariant differential forms $\rho_i \in \widetilde{\mathbf{\Omega}}^{r-i-1,i}_G$ such that

$$\begin{aligned}&d_{\mathcal{H}}\rho_1 = \gamma_1, \\ &d_{\mathcal{H}}\rho_2 + d_{\mathcal{V}}\rho_1 = \gamma_2, \\ &d_{\mathcal{H}}\rho_i + d_{\mathcal{V}}\rho_{i-1} + d_{\mathcal{W}}\rho_{i-2} = \gamma_i, \qquad i = 3, \ldots, r-1, \\ &d_{\mathcal{V}}\rho_{r-1} + d_{\mathcal{W}}\rho_{r-2} = \gamma_r.\end{aligned} \tag{6.2}$$

From (6.2) it follows that

$$d(\rho_1 + \rho_2 + \rho_3 + \cdots + \rho_{r-1}) = \gamma,$$

which proves that γ is d-exact. For $r = p + s$, the proof is similar except that now the condition $(\widetilde{\pi} \circ \widetilde{\pi}_{p,s})(\gamma) = 0$ implies, by the exactness of the rows (5.12), that the invariant type (p, s) component of γ is $d_{\mathcal{H}}$-exact. □

Theorem 6.2. The cohomology of the invariant Euler–Lagrange complex $\widetilde{\mathcal{E}}^*_G$

$$0 \longrightarrow \mathbb{R} \longrightarrow \widetilde{\mathbf{\Omega}}^{0,0}_G \xrightarrow{d_{\mathcal{H}}} \widetilde{\mathbf{\Omega}}^{1,0}_G \xrightarrow{d_{\mathcal{H}}} \cdots \xrightarrow{d_{\mathcal{H}}} \widetilde{\mathbf{\Omega}}^{p,0}_G \xrightarrow{\widetilde{E}} \widetilde{\mathcal{F}}^1_G \xrightarrow{\delta_{\mathcal{V}}} \widetilde{\mathcal{F}}^2_G \xrightarrow{\delta_{\mathcal{V}}} \cdots$$

is locally isomorphic to the invariant de Rham cohomology of J^∞.

Proof. Since the projection map $\widetilde{\pi}_{r,s} \colon \mathbf{\Omega}^{r+s}_G \to \widetilde{\mathbf{\Omega}}^{r,s}_G$ satisfies

$$\begin{aligned} \widetilde{\pi}_{r+1,0} \circ d &= d_{\mathcal{H}} \circ \widetilde{\pi}_{r,0}, && \text{for} \quad r \leq p-1, \\ \widetilde{\pi} \circ \widetilde{\pi}_{p,1} \circ d &= \widetilde{E} \circ \widetilde{\pi}_{p,0}, && \\ \widetilde{\pi} \circ \widetilde{\pi}_{p,s+1} \circ d &= \delta_{\mathcal{V}} \circ \widetilde{\pi} \circ \widetilde{\pi}_{p,s}, && \text{for} \quad s \geq 1, \end{aligned}$$

the map $\Psi \colon \mathbf{\Omega}^*_G \to \widetilde{\mathcal{E}}^*_G$ defined, for $\omega \in \mathbf{\Omega}^r_G$, by

$$\Psi(\omega) = \begin{cases} \widetilde{\pi}_{r,0}(\omega) & \text{for } r \leq p, \\ \widetilde{\pi} \circ \widetilde{\pi}_{p,s}(\omega) & \text{if } r = p+s \text{ and } s \geq 1, \end{cases} \tag{6.3}$$

is a cochain map. The induced map in cohomology will be denoted by $\Psi^* \colon H^*(\mathbf{\Omega}^*_G) \to H^*(\widetilde{\mathcal{E}}^*_G)$. The map Ψ^* is proved to be an isomorphism in cohomology by constructing the inverse map $\Phi \colon H^*(\widetilde{\mathcal{E}}^*_G) \to H^*(\mathbf{\Omega}^*_G)$. To define Φ we consider separately the two pieces of the complex $\widetilde{\mathcal{E}}^*_G$, beginning with the horizontal edge

$$0 \longrightarrow \mathbb{R} \longrightarrow \widetilde{\mathbf{\Omega}}^{0,0}_G \xrightarrow{d_{\mathcal{H}}} \widetilde{\mathbf{\Omega}}^{1,0}_G \xrightarrow{d_{\mathcal{H}}} \cdots \xrightarrow{d_{\mathcal{H}}} \widetilde{\mathbf{\Omega}}^{p,0}_G \xrightarrow{\widetilde{E}} \widetilde{\mathcal{F}}^1_G.$$

Let $[\omega] \in H^r(\widetilde{\mathcal{E}}^*_G)$ for $r \leq p$ and define $\omega_0 = \omega \in \widetilde{\mathbf{\Omega}}^{r,0}_G$. Using Theorem 5.10 and the differential relations (4.11) it is straightforward to find inductively $\omega_i \in \widetilde{\mathbf{\Omega}}^{r-i,i}_G$ such that

$$d_{\mathcal{H}}\omega_1 = -d_{\mathcal{V}}\omega_0, \qquad d_{\mathcal{H}}\omega_i = -d_{\mathcal{V}}\omega_{i-1} - d_{\mathcal{W}}\omega_{i-2}, \qquad 2 \leq i \leq r. \tag{6.4}$$

Let

$$\beta = \omega_0 + \omega_1 + \omega_2 + \cdots + \omega_r \ \in \ \mathbf{\Omega}^r_G. \tag{6.5}$$

The claim is that β is d-closed. The expression for $d\beta$ telescopes using the relations (6.4):

$$d\beta = \sum_{i=0}^{r} (d_{\mathcal{H}} + d_{\mathcal{V}} + d_{\mathcal{W}})\omega_i = d_{\mathcal{V}}\omega_r + d_{\mathcal{W}}\omega_{r-1},$$

where we have used the fact that $d_{\mathcal{W}}\omega_r = 0$. Using (4.11), one can verify that $d_{\mathcal{H}}(d_{\mathcal{V}}\omega_r + d_{\mathcal{W}}\omega_{r-1}) = 0$. Since $d_{\mathcal{V}}\omega_r + d_{\mathcal{W}}\omega_{r-1} \in \widetilde{\mathbf{\Omega}}^{0,r+1}_G$, by injectivity of $d_{\mathcal{H}} \colon \widetilde{\mathbf{\Omega}}^{0,r+1}_G \to \widetilde{\mathbf{\Omega}}^{1,r+1}_G$ it follows that $d\beta = d_{\mathcal{V}}\omega_r + d_{\mathcal{W}}\omega_{r-1} = 0$.

The cohomology class $[\beta] \in H^r(\boldsymbol{\Omega}_G^*)$ is independent of the choices taken for the ω_i. Indeed, any other $\overline{\omega}_i$ defined as in (6.4) must satisfy

$$\begin{aligned}
\overline{\omega}_0 &= \omega_0 + d_{\mathcal{H}}\alpha_0, \\
\overline{\omega}_1 &= \omega_1 + d_{\mathcal{H}}\alpha_1 + d_{\mathcal{V}}\alpha_0, \\
\overline{\omega}_i &= \omega_i + d_{\mathcal{H}}\alpha_i + d_{\mathcal{V}}\alpha_{i-1} + d_{\mathcal{W}}\alpha_{i-2}, \qquad 2 \le i \le r-1, \\
\overline{\omega}_r &= \omega_r + d_{\mathcal{V}}\alpha_{r-1} + d_{\mathcal{W}}\alpha_{r-2},
\end{aligned}$$

where $\alpha_i \in \widetilde{\boldsymbol{\Omega}}_G^{r-i-1,i}$. Hence, defining $\overline{\beta}$ as in (6.5) we obtain

$$\overline{\beta} = \beta + d(\alpha_0 + \alpha_1 + \cdots + \alpha_{r-1}).$$

Thus the map Φ may be defined by $\Phi([\omega]) = [\beta]$.

It now follows that Ψ^* and Φ are mutually inverse. First observe that for $[\omega] \in H^r(\widetilde{\boldsymbol{\Omega}}_G^{*,0})$, we have $\Psi^* \circ \Phi([\omega]) = \Psi^*([\beta]) = [\omega]$. Next, let $\alpha \in \boldsymbol{\Omega}_G^r$ be a d-closed form and let $\alpha_0 = \widetilde{\pi}^{r,0}(\alpha)$. Since $d\alpha = 0$ it follows that $d_{\mathcal{H}}\alpha_0 = 0$, hence we may define inductively, starting with $\alpha_0 \in \widetilde{\boldsymbol{\Omega}}_G^{r,0}$, a $\beta \in \boldsymbol{\Omega}_G^r$ as in (6.5). Then $\Phi \circ \Psi^*([\alpha]) = \Phi([\alpha_0]) = [\beta]$. Since $\widetilde{\pi}_{r,0}(\alpha) = \widetilde{\pi}_{r,0}(\beta) = \alpha_0$, the difference $\beta - \alpha$ satisfies the hypotheses of Lemma 6.1 and is thus d-exact. Hence $[\beta] = [\alpha]$.

The case $r = p + s$, $s \ge 1$, corresponding to the second piece of the complex,

$$\widetilde{\mathcal{F}}_G^1 \xrightarrow{\delta_{\mathcal{V}}} \widetilde{\mathcal{F}}_G^2 \xrightarrow{\delta_{\mathcal{V}}} \widetilde{\mathcal{F}}_G^3 \xrightarrow{\delta_{\mathcal{V}}} \cdots$$

is dealt with very similarly. The condition $\delta_{\mathcal{V}}\omega_0 = 0$ for $\omega_0 \in \widetilde{\mathcal{F}}_G^s$ implies that there is some $\omega_1 \in \widetilde{\boldsymbol{\Omega}}_G^{p-1,s+1}$ such that $d_{\mathcal{H}}\omega_1 = -d_{\mathcal{V}}\omega_0$. Setting $\beta = \omega_0 + \omega_1 + \cdots + \omega_p$, where $\omega_i \in \widetilde{\boldsymbol{\Omega}}_G^{p-i,s+i}$ is defined inductively via the relation $d_{\mathcal{H}}\omega_i = -d_{\mathcal{W}}\omega_{i-2} - d_{\mathcal{V}}\omega_{i-1}$, $i = 2, \ldots, p$, we obtain the inverse Φ to Ψ^* just as in the previous argument. □

7 Lie Algebra Cohomology

Definition 7.1. Let G be a connected r-dimensional Lie group with Lie algebra $\mathfrak{g}$. The *Lie algebra cohomology* $H^*(\mathfrak{g})$ is the de Rham cohomology of the complex of invariant differential forms on G.

We remark that the de Rham complex of invariant differential forms on G and the complex $(\Lambda^r(\mathfrak{g}), d)$ of alternating multilinear functionals on $\mathfrak{g}$ with

$$d\alpha(X_0, \ldots, X_r) = \sum_{i \le j} (-1)^{i+j} \alpha\big([X_i, X_j], X_0, \ldots, \widehat{X_i}, \ldots, \widehat{X_j}, \ldots, X_r\big),$$

appearing in many references, are isomorphic.

We now construct a local isomorphism between the G-invariant de Rham complex on M and the Lie algebra cohomology for $\mathfrak{g}$. The construction of this isomorphism roughly follows [5], with the added computational and conceptual advantage of moving frames.

Theorem 7.2. If $z_0 \in M$ is a regular point of the group action G, then there is a neighborhood $\mathcal{U} \subset M$ of z_0 such that $H^*(\boldsymbol{\Omega}_G^*(\mathcal{U})) \simeq H^*(\mathfrak{g})$.

Proof. By Theorem 3.2 there is a neighborhood $\mathcal{V}$ of z_0 and a moving frame $\rho\colon \mathcal{V} \to G$ corresponding to a cross-section $\mathcal{K} \subset \mathcal{V}$. Restrict to a neighborhood $\mathcal{U} \subset \mathcal{V}$ so that there is a strong deformation retract $H(z,t)$ of $\mathcal{K} \cap \mathcal{U}$ to z_0 and such that the expression

$$\rho(z)^{-1} \cdot H(\rho(z) \cdot z, t) \tag{7.1}$$

is defined for all $z \in \mathcal{U}$. This can be done for instance by introducing flat local coordinates on M which identify a neighborhood $\mathcal{U} \subset \mathcal{V}$ of z_0 with $G_0 \times \mathcal{K}$, where G_0 is a suitable neighborhood of the identity in G, [14]. The map (7.1) defines an equivariant strong deformation retract of $\mathcal{U}$ onto the group orbit $\mathcal{O}$ of z_0 in $\mathcal{U}$. Thus the invariant de Rham cohomology of the neighborhood $\mathcal{U}$ is isomorphic to that of its submanifold $\mathcal{O}$: $H^*(\mathbf{\Omega}^*_G(\mathcal{U})) \simeq H^*(\mathbf{\Omega}^*_G(\mathcal{O}))$.

Now, let $\mu^1, \ldots, \mu^r$, be a basis of Maurer–Cartan forms for G and let $\nu^1 = \rho^*(\mu^1)$, $\ldots$, $\nu^r = \rho^*(\mu^r)$ be the pull-backs of the Maurer–Cartan forms via the moving frame. The forms ν^i are invariant one-forms on M whose restrictions $\nu^i|_{\mathcal{O}}$ form an invariant coframe on $\mathcal{O}$ and hence generate the invariant de Rham complex on $\mathcal{O}$. Furthermore, since pullback commutes with d, the structure equations for the forms ν^i are the same as the Maurer–Cartan structure equations. Hence the moving frame pullback provides an isomorphism of the complex of invariant differential forms on G and the invariant de Rham complex on $\mathcal{O}$, which is in turn isomorphic to the invariant de Rham complex on $\mathcal{U}$. $\square$

Under our regularity assumption that G acts effectively on subsets, the prolonged transformation group will act locally freely on an open subset of J^n for n sufficiently large, [14]. Then the following corollary is a direct consequence of Theorem 7.2.

Corollary 7.3. Let G be a Lie group acting on M. Suppose that $z^{(\infty)} \in \mathrm{J}^\infty$ is a regular jet of the prolonged group action $G^{(\infty)}$. Then there is a neighborhood $\mathcal{U}^\infty \subset \mathcal{V}^\infty \subset \mathrm{J}^\infty$ of $z^{(\infty)}$ such that

$$H^*(\mathbf{\Omega}^*_G(\mathcal{U}^\infty)) \simeq H^*(\mathfrak{g}^*).$$

Combining Corollary 7.3 and Theorem 6.2, we obtain the main result of the paper.

Theorem 7.4. Let G be a Lie group acting on M. Suppose that $z^{(\infty)} \in \mathrm{J}^\infty$ is a regular point of the prolonged action $G^{(\infty)}$. Then there is a neighborhood $\mathcal{U}^\infty \subset \mathcal{V}^\infty \subset \mathrm{J}^\infty$ of $z^{(\infty)}$ such that

$$H^*(\widetilde{\mathcal{E}}^*(\mathcal{U}^\infty)) \simeq H^*(\mathfrak{g}^*).$$

To proceed further we extend the definition of the non-invariant boundary operators (2.7) to allow arbitrary $p+s$ forms. Given a differential form $\Omega \in \mathbf{\Omega}^{p+s}$, with $s \geq 0$, the *extended boundary operator* is

$$\delta^*_V(\Omega) = \pi \circ \pi_{p,s} \circ d_V(\Omega) = \pi \circ \pi_{p,s} \circ d(\Omega). \tag{7.2}$$

A property of the extended boundary operator δ^*_V is that it annihilates all components in Ω which are not of maximal horizontal degree. The introduction of the extended boundary operator (7.2) first appeared in [20] and was used to define the *extended Euler derivative.*

Lemma 7.5. Let $\Omega, \Psi \in \mathbf{\Omega}^{p+s}$. If $\pi_{p,s}(\Omega) = \pi_{p,s}(\Psi)$ then $\delta^*_V(\Omega) = \delta^*_V(\Psi)$.

Lemma 7.6. Let $\widetilde{\Omega} \in \widetilde{\mathbf{\Omega}}^{p,s}_G$ and $\Omega = \pi_{p,s}(\widetilde{\Omega}) \in \mathbf{\Omega}^{p,s}$, then

$$\delta_V^*(\Omega) = \pi_{p,s+1} \circ \delta_{\mathcal{V}}(\widetilde{\Omega}).$$

Proof. By Lemma 7.5

$$\delta_V^*(\Omega) = \delta_V^*(\widetilde{\Omega}) = \pi \circ \pi_{p,s+1} \circ d(\widetilde{\Omega}) = \pi \circ \pi_{p,s+1}(d_{\mathcal{H}}\widetilde{\Omega} + d_{\mathcal{V}}\widetilde{\Omega} + d_{\mathcal{W}}\widetilde{\Omega}).$$

The first and third terms in the last equality vanish since $d_{\mathcal{H}}\widetilde{\Omega} = 0$ as $\widetilde{\Omega}$ is of maximal invariant horizontal degree and $d_{\mathcal{W}}\widetilde{\Omega} \in \widetilde{\mathbf{\Omega}}^{p-1,s+2}$ which implies that $\pi_{p,s+1}(d_{\mathcal{W}}\widetilde{\Omega}) = 0$. Thus we are left with

$$\delta_V^*(\Omega) = \pi_{p,s+1} \circ \widetilde{\pi} \circ d_{\mathcal{V}}(\widetilde{\Omega}) = \pi_{p,s+1} \circ \delta_{\mathcal{V}}(\widetilde{\Omega}).$$

□

Theorem 7.4 combined with Lemma 7.6 gives a cohomological condition for the solution to the invariant inverse problem of variational calculus.

Corollary 7.7. Let $\mathcal{U}^\infty$ be as in Theorem 7.4 and suppose that $H^{p+1}(\mathfrak{g}^*) = 0$. Then every G-invariant source form on $\mathcal{U}^\infty$ which is the Euler–Lagrange form of some Lagrangian is the Euler–Lagrange form of a G-invariant Lagrangian.

8 Examples

In this section we consider the geometry of Euclidean and equi-affine curves in the plane and Euclidean surfaces in $\mathbb{R}^3$ to illustrate the Theorems discussed in Sections 6 and 7.

Example 8.1. We first consider our running example of the Euclidean group $SE(2)$. The Maurer–Cartan structure equations for this group are

$$d\mu^1 = \mu^2 \wedge \mu^3, \qquad d\mu^2 = -\mu^1 \wedge \mu^3, \qquad d\mu^3 = 0,$$

where

$$\mu^1 = da + bd\phi, \qquad \mu^2 = db - ad\phi, \qquad \mu^3 = d\phi. \tag{8.1}$$

It follows that the non-trivial[3] cohomology classes of $H^*(\mathfrak{se}^*(2))$ are

$$[\mu^3], \qquad [\mu^1 \wedge \mu^2], \qquad [\mu^1 \wedge \mu^2 \wedge \mu^3]. \tag{8.2}$$

Taking the pull-backs of the Maurer–Cartan forms (8.1) by the moving frame (3.5) leads to the invariant one-forms

$$\nu^1 = -\frac{dx + u_x du}{(1+u_x^2)^{1/2}}, \qquad \nu^2 = \frac{u_x dx - du}{(1+u_x^2)^{1/2}}, \qquad \nu^3 = -\frac{du_x}{1+u_x^2}.$$

The pull-backs of the cohomology classes (8.2) give the invariant de Rham cohomology classes

$$[\kappa\varpi + \vartheta_1], \qquad [\varpi \wedge \vartheta], \qquad [\varpi \wedge \vartheta \wedge \vartheta_1]. \tag{8.3}$$

[3]We neglect the trivial cohomology class from our considerations.

Applying the map (6.3) to the cohomology classes (8.3) we find that the non-trivial cohomology classes of the invariant Euler–Lagrange complex are

$$[\kappa\varpi], \qquad [\varpi\wedge\vartheta], \qquad [\varpi\wedge\vartheta\wedge\vartheta_1]. \tag{8.4}$$

We now show that (8.4) is related to the cohomology classes obtained in [6], where parametrized curves

$$z:\mathbb{R}\to\mathbb{R}^2, \qquad z(t)=(x(t),u(t))$$

are considered. In this setting, the natural group action to consider is the infinite-dimensional Lie pseudo-group

$$\overline{G}=\mathrm{Diff}(\mathbb{R})^+\times SE(2), \qquad (\psi,R,b)\cdot(t,z)=(\psi(t),Rz+b), \tag{8.5}$$

where ψ is a local diffeomorphism of $\mathbb{R}$ with $\psi'(t)>0$, $R\in SO(2)$ and $b\in\mathbb{R}^2$. Under the pseudo-group action (8.5), the invariant Euler–Lagrange complex has four non-trivial cohomology classes, [6], and three of the four originate from the cohomology of $SE(2)$. These are given by

$$\begin{aligned}
&[\lambda]=[\kappa\omega]=[\frac{\dot x\ddot u-\ddot x\dot u}{\dot x^2+\dot u^2}dt], \qquad [\delta]=[(\dot u dx-\dot x du)\wedge dt], \\
&[\beta]=\left[\omega\wedge\left(\kappa dx\wedge du+\frac{\dot u^2 dx\wedge d\dot x-\dot x\dot u(dx\wedge d\dot u+du\wedge d\dot x)+\dot x^2 du\wedge d\dot u}{(\dot x^2+\dot u^2)^{3/2}}\right)\right],
\end{aligned} \tag{8.6}$$

where $\omega=\sqrt{\dot x^2+\dot u^2}\,dt$ is the arc length form. To recover (8.6) from (8.4) we first observe that when $(x,u)=(x(t),u(t))$

$$\begin{aligned}
\theta&=du-u_x dx=\frac{x_t du-u_t dx}{x_t}, \\
\theta_1&=du_x-u_{xx}dx=\frac{x_t du_t-u_t dx_t}{x_t^2}-\frac{u_{tt}x_t-u_t x_{tt}}{x_t^3}dx, \\
\varpi&=\sqrt{x_t^2+u_t^2}dt+\frac{dx-x_t dt}{x_t}+\frac{u_t\theta}{\sqrt{x_t^2+u_t^2}}.
\end{aligned} \tag{8.7}$$

Next, let $\overline{\mathbf{\Omega}}^{r,s}$ be the bundle of (r,s)-forms generated by the horizontal form dt and the basic contact forms

$$\theta^x_k=dx_k-x_{k+1}dt, \qquad \theta^u_k=du_k-u_{k+1}dt, \qquad k\geq 0.$$

If we denote by $\overline{\pi}_{r,s}\colon\mathbf{\Omega}\to\overline{\mathbf{\Omega}}^{r,s}$ the projection onto $\overline{\mathbf{\Omega}}^{r,s}$, then we have the equalities

$$[\overline{\pi}_{1,0}(\kappa\varpi)]=[\lambda], \qquad [\overline{\pi}_{1,1}(\varpi\wedge\vartheta)]=[\delta], \qquad [\overline{\pi}_{1,2}(\varpi\wedge\vartheta\wedge\vartheta_1)]=[\beta].$$

Example 8.2. A more substantial example is provided by the geometry of equi-affine planar curves, [15]. The equi-affine group $SA(2)=SL(2)\ltimes\mathbb{R}^2$ acts on $M=\mathbb{R}^2$ as area-preserving affine transformation

$$g\cdot(x,u)=(X,U)=(\alpha x+\beta u+a,\gamma x+\delta u+b), \qquad \alpha\delta-\beta\gamma=1.$$

The coordinate cross-section $X=U=U_X=0$, $U_{XX}=1$, $U_{XXX}=0$, leads to the classical equi-affine moving frame, [13, 20],

$$\begin{aligned}
&a=\frac{x(u_x u_{xxx}-3u_{xx}^2)-uu_{xxx}}{3u_{xx}^{5/3}}, \qquad b=\frac{xu_x-u}{u_{xx}^{1/3}}, \\
&\alpha=\frac{3u_{xx}^2-u_x u_{xxx}}{3u_{xx}^{5/3}}, \qquad \delta=\frac{1}{u_{xx}^{1/3}}, \qquad \beta=\frac{u_{xxx}}{3u_{xx}^{5/3}}, \qquad \gamma=-\frac{u_x}{u_{xx}^{1/3}}.
\end{aligned} \tag{8.8}$$

The fundamental differential invariant is the equi-affine curvature

$$\kappa = \iota(u_{xxxx}) = \frac{3u_{xx}u_{xxxx} - 5u_{xxx}^2}{3u_{xx}^{8/3}}.$$

The corresponding invariant horizontal form is

$$\varpi = \iota(dx) = u_{xx}^{1/3}\,dx + \frac{u_{xxx}}{3u_{xx}^{5/3}}\,\theta,$$

while the invariant contact forms are

$$\vartheta = \frac{\theta}{u_{xx}^{1/3}}, \qquad \vartheta_1 = \frac{3u_{xx}\theta_x - u_{xxx}\theta}{3u_{xx}^{5/3}},$$
$$\vartheta_2 = \frac{u_{xx}\theta_{xx} - u_{xxx}\theta_x}{u_{xx}^2}, \qquad \vartheta_3 = \frac{3u_{xx}^2\theta_{xxx} - 6u_{xx}u_{xxx}\theta_{xx} + u_{xxx}^2\theta_x - \kappa u_{xx}^{5/3}u_{xxx}\theta}{3u_{xx}^{10/3}},$$

and so on. A basis of Maurer–Cartan forms for $SA(2)$ is given by

$$\begin{gathered}\mu^1 = da + (\beta b - \delta a)d\alpha + (\gamma a - \alpha b)d\beta, \qquad \mu^2 = db + (\beta b - \delta a)d\gamma + (\gamma a - \alpha b)d\delta, \\ \mu^3 = \delta d\alpha - \gamma d\beta, \qquad \mu^4 = \alpha d\beta - \beta d\alpha, \qquad \mu^5 = \delta d\gamma - \gamma d\delta,\end{gathered} \tag{8.9}$$

where $\delta d\alpha + \alpha d\delta - \beta d\gamma - \gamma d\beta = 0$. The corresponding Maurer–Cartan structure equations are

$$\begin{gathered}d\mu^1 = \mu^4 \wedge \mu^2 + \mu^3 \wedge \mu^1, \qquad d\mu^2 = \mu^5 \wedge \mu^1 + \mu^2 \wedge \mu^3, \\ d\mu^3 = \mu^4 \wedge \mu^5, \qquad d\mu^4 = 2\,\mu^3 \wedge \mu^4, \qquad d\mu^5 = 2\,\mu^5 \wedge \mu^3.\end{gathered} \tag{8.10}$$

From (8.10) we conclude that the non-trivial Lie algebra cohomology classes are

$$[\mu^1 \wedge \mu^2], \qquad [\mu^3 \wedge \mu^4 \wedge \mu^5], \qquad [\mu^1 \wedge \mu^2 \wedge \mu^3 \wedge \mu^4 \wedge \mu^5]. \tag{8.11}$$

Taking the pull-back of the Maurer–Cartan forms (8.9) by the moving frame (8.8) we obtain the invariant one-forms

$$\rho^*(\mu^1) = -\varpi, \qquad \rho^*(\mu^2) = -\vartheta,$$
$$\rho^*(\mu^3) = \frac{\vartheta_2}{3}, \qquad \rho^*(\mu^4) = \frac{\kappa\varpi + \vartheta_3}{3}, \qquad \rho^*(\mu^5) = -(\varpi + \vartheta_1).$$

Thus the pull-back of the cohomology classes (8.11) gives the three invariant de Rham cohomology classes

$$[\varpi \wedge \vartheta], \qquad [\vartheta_1 \wedge \vartheta_2 \wedge \vartheta_3 + \kappa\varpi \wedge \vartheta_1 \wedge \vartheta_2 + \varpi \wedge \vartheta_2 \wedge \vartheta_3], \qquad [\varpi \wedge \vartheta \wedge \vartheta_1 \wedge \vartheta_2 \wedge \vartheta_3]. \tag{8.12}$$

The cohomology classes of the invariant Euler–Lagrange complex are obtained by applying the map (6.3) to (8.12). Consequently, the non-trivial $SA(2)$-invariant local Euler–Lagrange cohomology classes for equi-affine planar curves are

$$[\varpi \wedge \vartheta], \qquad [\kappa\varpi \wedge \vartheta_1 \wedge \vartheta_2 + \varpi \wedge \vartheta_2 \wedge \vartheta_3], \qquad [\varpi \wedge \vartheta \wedge \vartheta_1 \wedge \vartheta_2 \wedge \vartheta_3].$$

Example 8.3. As a final example we consider the action of $SE(3) = SO(3) \ltimes \mathbb{R}^3$ on surfaces in $\mathbb{R}^3$ (with coordinates (x, y, u)) given by the infinitesimal generators

$$\mathbf{v}_1 = x\partial_y - y\partial_x, \quad \mathbf{v}_2 = y\partial_u - u\partial_y, \quad \mathbf{v}_3 = u\partial_x - x\partial_u, \quad \mathbf{v}_4 = \partial_x, \quad \mathbf{v}_5 = \partial_y, \quad \mathbf{v}_6 = \partial_u.$$

Let $\mu^1, \dots \mu^6$ be a basis of Maurer–Cartan forms dual to (the Lie algebra basis corresponding to) the infinitesimal generators. The corresponding structure equations are

$$d\mu^1 = \mu^2 \wedge \mu^3, \quad d\mu^2 = -\mu^1 \wedge \mu^3, \quad d\mu^3 = \mu^1 \wedge \mu^2, \quad d\mu^4 = -\mu^1 \wedge \mu^5 + \mu^3 \wedge \mu^6,$$
$$d\mu^5 = \mu^1 \wedge \mu^4 - \mu^2 \wedge \mu^6, \qquad d\mu^6 = \mu^2 \wedge \mu^5 - \mu^3 \wedge \mu^4.$$

A straightforward computation using Maple shows that the non-trivial Lie algebra cohomology classes are

$$[\mu^1 \wedge \mu^2 \wedge \mu^3], \qquad [\mu^4 \wedge \mu^5 \wedge \mu^6], \qquad [\mu^1 \wedge \mu^2 \wedge \mu^3 \wedge \mu^4 \wedge \mu^5 \wedge \mu^6]. \tag{8.13}$$

Unlike the previous examples an explicit formula for the moving frame is not given here, but instead the cross-section

$$X = 0, \qquad Y = 0, \qquad U = 0, \qquad U_X = 0, \qquad U_Y = 0, \qquad U_{XY} = 0,$$

and the recurrence relation (4.10) are used to express the moving frame pull-backs $\nu^1, \dots, \nu^6$ of the Maurer–Cartan forms in terms of known invariants. The computations hold for non-umbilic points, i.e. $\kappa^1 \neq \kappa^2$, and yield

$$\nu^1 = \frac{\kappa^1_{,2}\varpi^1 + \kappa^2_{,1}\varpi^2 + \vartheta_{12}}{\kappa^2 - \kappa^1}, \qquad \nu^2 = -\kappa^2\varpi^2 - \vartheta_2,$$
$$\nu^3 = \kappa^1\varpi^1 + \vartheta_1, \qquad \nu^4 = -\varpi^1, \qquad \nu^5 = -\varpi^2, \qquad \nu^6 = -\vartheta,$$

where

$$\kappa^1 = \iota(u_{xx}), \qquad \kappa^2 = \iota(u_{yy}), \qquad \varpi^1 = \iota(dx), \qquad \varpi^2 = \iota(dy),$$
$$\vartheta_J = \iota(\theta_J), \qquad d_{\mathcal{H}}\kappa^i = \kappa^i_{,1}\varpi^1 + \kappa^i_{,2}\varpi^2.$$

Here κ^1 and κ^2 are the principal curvatures of the surface and $\kappa^i_{,1}$, $\kappa^i_{,2}$ denote their invariant derivatives. These computations illustrate the ability to compute *intrinsically*, i.e. without coordinate expressions for the moving frame, normalized invariants, or pulled-back Maurer–Cartan forms. See [28] for more details. It follows that the pull-back of the Lie algebra cohomology classes (8.13) by the moving frame gives the invariant de Rham cohomology classes

$$\begin{aligned}\Bigg[\frac{1}{\kappa^2 - \kappa^1}\bigg(& -\kappa^1_{,2}\kappa^2\varpi^1 \wedge \varpi^2 \wedge \vartheta_1 - \kappa^2_{,1}\kappa^1\varpi^1 \wedge \varpi^2 \wedge \vartheta_2 + \kappa^1_{,2}\varpi^1 \wedge \vartheta_1 \wedge \vartheta_2 \\ & + \kappa^2_{,1}\varpi^2 \wedge \vartheta_1 \wedge \vartheta_2 + \kappa^1\kappa^2\varpi^1 \wedge \varpi^2 \wedge \vartheta_{12} - \kappa^2\varpi^2 \wedge \vartheta_1 \wedge \vartheta_{12} \\ & + \kappa^1\varpi^1 \wedge \vartheta_2 \wedge \vartheta_{12} + \vartheta_1 \wedge \vartheta_2 \wedge \vartheta_{12}\bigg)\Bigg],\end{aligned}$$
$$[\varpi^1 \wedge \varpi^2 \wedge \vartheta] \qquad \text{and} \qquad [\varpi^1 \wedge \varpi^2 \wedge \vartheta \wedge \vartheta_1 \wedge \vartheta_2 \wedge \vartheta_{12}].$$

Applying the map (6.3) gives the corresponding invariant Euler–Lagrange cohomology classes

$$\left[\frac{-\kappa^1_{,2}\kappa^2\varpi^1 \wedge \varpi^2 \wedge \vartheta_1 - \kappa^2_{,1}\kappa^1\varpi^1 \wedge \varpi^2 \wedge \vartheta_2 + \kappa^1\kappa^2\varpi^1 \wedge \varpi^2 \wedge \vartheta_{12}}{\kappa^2 - \kappa^1}\right],$$
$$[\varpi^1 \wedge \varpi^2 \wedge \vartheta] \qquad \text{and} \qquad [\varpi^1 \wedge \varpi^2 \wedge \vartheta \wedge \vartheta_1 \wedge \vartheta_2 \wedge \vartheta_{12}].$$

9 Conclusion

Using the method of moving frames we have been able to extend the results of [5] to non-projectable group actions. Note that we recover the results of Anderson and Pohjanpelto if the group action is projectable. Indeed, for such group actions the invariant bigrading $\widetilde{\mathbf{\Omega}}^{*,*}$ is equal to the noninvariant bigrading $\mathbf{\Omega}^{*,*}$, the projection maps (4.7) are equal to the identity map, the differential $d_{\mathcal{W}}$ is identically zero, and the bundle of vertical vector fields $\mathbf{V}$ is equal to the bundle of invariant vertical vector fields $\mathbf{V}_G$.

As illustrated in the third example, the computation of the Euler–Lagrange cohomology classes may be done intrinsically, i.e. without coordinate expressions for the moving frame and Maurer–Cartan forms. The only data needed is the choice of a cross-section, the infinitesimal symmetry generators, the recurrence relation (4.10) and the Lie algebra cohomology classes which can easily be obtained using a symbolic software.

Finally, applications of our results to the geometry of higher dimensional submanifolds is of interest and tractable with benefit of intrinsic computation. Also, the similarity of Olver and Pohjanpelto's new method of moving frames for Lie pseudo-groups and the moving frame theory for finite-dimensional Lie groups [29, 30] allows the techniques of this paper to be extended to infinite-dimensional Lie pseudo-group actions.

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