POLARIZED HARMONIC MAPPINGS AND OPTIMAL MOVING FRAMES

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Abstract. We introduce the notion of a polarized harmonic mapping $M \to N$ of Riemannian manifolds and its infinitesimal analogue, a polarized harmonic connection. We study the integrability of these polarized harmonic connections when M is foliated by the action of a Lie group G. In the case that M is a Kähler manifold and G has dimension 1, the polarized harmonic connection is integrable and we obtain a polarized harmonic mapping $M \to G$, known as an optimal moving frame (or optimal G-frame). These ideas are illustrated using the dynamical system of N-point vortices in the plane.

Contents

1. Notation

M	n-dimensional C^∞-manifold
$\mathcal{F}$	smooth foliation on M
u	smooth vector field on M
G	k-dimensional real Lie group
$\mathfrak{g}$	Lie algebra of a Lie group G
$\Phi : G \times M \to M$	a smooth action of G on M
ξ_M	vector field on M induced by the action of $\xi \in \mathfrak{g}$
$\mathfrak{g}_m$ or $T_m(Gm)$	the tangent space to the G-orbit of the point $m \in M$
$\phi_* : T(M) \to T(N)$	tangent mapping of the smooth mapping $\phi : M \to N$

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2. Introduction

It happens quite often that in the phase space M of a dynamical system (M, u) there is a natural action of a Lie group G. This group may be a symmetry group of the dynamical system or have some weaker relation to it, e.g. the phase flow of the vector field u maps orbits of G into orbits. Alternatively, G may be interesting from a geometric point of view, as the isometry group of some metric on M, for example.

In some cases it is useful to present the dynamical vector field u as a sum of motion along G-orbits and motion transversal to them, i.e. to decompose the vector field u into a component along G and a complementary component:

$$u(m) = u_G(m) + u_r(m). \tag{2.1}$$

There are many examples of such a decomposition, from the removal of center of mass motion in mechanics to the decomposition of dynamics in terms of different degrees of freedom (translational, rotational, oscillatory, etc.) in nuclear physics. Geometric procedures have been developed in order to make such a decomposition in a regular way. Examples include Hamiltonian reduction, Lagrange reduction and collective modes, [5, 9, 13].

The infinitesimal decomposition (2.1) requires a smooth distribution H in M transversal to the orbits of G and complementary to the tangent subspaces $\mathfrak{g}_m = T(Gm)$. This distribution yields a decomposition of the tangent bundle

$$T_m(M) = \mathfrak{g}_m \oplus H_m.$$

If the projection mapping $M \to M/G$ is a fibration, a choice of H defines an Ehresmann connection on this bundle. If G acts with a single orbit type G/K, one way to construct such a connection is to define a smooth projection $\phi : M \to G/K$ of maximal rank along orbits (for example G-equivariant) and to take $H_m = \ker(\phi_{*m})$. Such mappings ϕ present a natural generalization of G-frames (see Definition 3).

When M is the phase space of a dynamical system (M, u), it is natural to consider the problem of characterizing connections which are extremal or optimal with respect to u in some sense, measured by an energy functional. One way to define an optimal connection is to take a mapping $\phi : M \to G/K$ which is *maximal* in the direction of u and *minimal* in transversal directions. Centroidal frames, [9, 10] are an example of such optimal connections, where energy of relative motion is minimized. In what follows, we define *polarized harmonic mappings* as mappings which maximize a *polarized energy* and study these mappings and their infinitesimal versions, *polarized connections*.

In Section 3 we introduce background ideas and define the polarized energy function for a vector field u on a foliated Riemannian manifold M. In Section 4 we introduce the polarized connection defined by the foliation $\mathcal{F}$ on M. We prove in Theorem 1, that in a neighborhood of a point of locally constant rank a unique polarized connection exists and is expressed by an explicit formula. In Section 5 we study the integrability of the polarized connection in the case where the foliation $\mathcal{F}$ is defined by a free action of a Lie group G. We introduce the notion of a polarized harmonic mapping $\phi : M \to N$ of Riemannian manifolds in Section 6. In Section 7 we discuss polarized harmonic mappings in the case where $N = G$, a compact Lie group, M is an almost Kähler manifold, and u is a Hamiltonian system with a harmonic Hamiltonian function H. When G has dimension 1, the polarized connection is integrable and there is one point of maximum polarized energy. In Section 8 we describe the dynamical system of point vortices on the plane and show that the polarized connection for rotational symmetry is integrable and coincides with the centroidal frame found in [9]. Finally, in Section 9 we study the Euler-Lagrange equations for the polarized energy functional. We show that the Euler–Lagrange equations are hyperbolic with propagation in the direction of the vector field u.

3. Transversal connections on foliated manifolds

In this section we introduce and discuss Ehresmann connections related to a foliation on a manifold, following [5, 14]. Let $\mathcal{F}$ be a k-dimensional smooth foliation of a manifold M. This means that each leaf $\mathcal{F}_m$ of the foliation is a k-dimensional smooth submanifold of M, different leaves do not intersect, and M is the union of the leaves $\mathcal{F}_m$. An important example is the foliation by orbits of a left action of a Lie group G on M, provided this action has only one orbit type, say G/K. Each leaf $\mathcal{F}_m = Gm$ is the G-orbit of a point $m \in M$. We will call such foliations G-orbit foliations. If G acts freely on M, each orbit has type G. If all orbits are closed, the space of orbits M/G carries the structure of manifold and we have a principal bundle $M \to M/G$ with structure group G.

In general, a foliation $\mathcal{F}$ generates the exact sequence of vector bundles

$$0 \to T\mathcal{F} \to T(M) \to N(M) = T(M)/T\mathcal{F} \to 0, \tag{3.1}$$

where $N(M)$ is called the *normal bundle* of the foliation.

Definition 1. *A transversal connection on M associated with the foliation $\mathcal{F}$ is a splitting of the sequence of vector bundles* (3.1) *over M.*

We will usually refer to transversal connection simply as a connection. There are several ways to specify a connection.

Lemma 1. *The following objects are equivalent:*

1. *A splitting of the sequence of vector bundles* (3.1)*.*
2. *A (vertical) projection $v_m : T_m(M) \to T_m(\mathcal{F}_m)$ at each point $m \in M$, depending smoothly on m.*
3. *A smooth distribution $\mathcal{H}$ of (horizontal) subspaces H_m of $T_m(M)$ complementary to the (vertical) subspaces $V_m = T\mathcal{F}_m$, giving the splitting $T_m(M) = V_m \oplus H_m$.*
4. *A $T\mathcal{F}$-valued differential 1-form ω on M such that for all $\xi \in V_m$, $\omega_m(\xi) = \xi$.*

Definition 2. *If $\mathcal{F}$ is a G-orbit foliation, a connection ω satisfying the condition $g^*\omega = \omega$ (or, equivalently $g_* H_m = H_{gm}$) for $g \in G$ is called an equivariant connection.*

Denote by $Con_{\mathcal{F}}(M)$ the set of all the connections defined by the foliation $\mathcal{F}$ and by $Con_G(M)$ the equivariant connections defined by a G-orbit foliation.

Definition 3. *Suppose G acts smoothly on a manifold M with one type of orbits, G/K, where K is a closed subgroup of G. A smooth mapping*

$$\phi : M \to G/K$$

of maximal rank at all points $m \in M$ will be called a G-frame on M. If ϕ is G-equivariant (i.e. satisfies $\phi(gm) = g\phi(m)$), ϕ will be called an equivariant G-frame.

Let $\mathcal{F}$ be a G-orbit foliation. A G-frame on M defines transversal connection ω_ϕ via the assignment of horizontal subspaces

$$H_m = \ker(\phi_{*m}).$$

For an equivariant G-frame ϕ, the corresponding connection ω_ϕ is equivariant.

Lemma 2. *Suppose that G is connected and acts freely on M. Let $\phi : M \to G$ be an equivariant G-frame. Then M is equivariantly isomorphic to the direct product $G \times \Gamma$ where $\Gamma = \phi^{-1}(e)$.*

Proof. Let $\Gamma = \phi^{-1}(e)$. By the local linearity of epimorphisms, Γ is a smooth submanifold of M. For any $g \in G$, $\phi^{-1}(g) = g\phi^{-1}(e) = g\Gamma$ so that action by g^{-1} defines a diffeomorphism of Γ and $\phi^{-1}(g)$. Define the mapping $G \times \Gamma \to M$ by $(g, m) \to gm$. This mapping is a smooth equivariant bijection of manifolds. □

It follows from Lemma 2 that the projection $\pi : M \to \tilde{M} = M/G$ is a trivial principal G-bundle. Any global section $s : \tilde{M} \to M$ of this bundle defines a corresponding equivariant G-frame $\phi_s(gs(x)) = g$ for all $x \in \tilde{M}, g \in G$. An equivariant G-frame is defined by the pre-image Γ of the unit of G: $\Gamma = \phi^{-1}(e) = s(\tilde{M})$. Γ is a submanifold of M, and it coincides with the image of the corresponding global section $s \in \Gamma(\pi)$. For any $h \in G$, $\phi^{-1}(h) = h\Gamma$.

Fix a section s and corresponding equivariant G-frame ϕ and submanifold Γ. Any point $m \in M$ has a unique representation

$$m = \phi(m)k_\phi(m),\ k_\phi(m) \in \Gamma.$$

Any other equivariant G-frame ψ is defined by its submanifold Γ_ψ and corresponding section s_ψ. These are defined by the mapping $\chi : \tilde{M} \to G$ for which

$$s_\psi(x) = \chi(x)s(x),\ \Gamma_\psi = \{\chi(\pi(m))m | m \in \Gamma\},\ \psi(m) = \phi(m)\chi(\pi(m))^{-1}.$$

The last equality follows from the representation

$$m = \phi(m)k_\phi(m) = \phi(m)\chi(\pi(m))^{-1}\chi(\pi(m))k_\psi(m).$$

As a result we have proved the following Lemma.

Lemma 3. *Fix a section $s \in \Gamma(\pi)$, corresponding equivariant G-frame ϕ and submanifold Γ. The mapping $\tilde{M}^G \to Con_G(M)$ defined by $\chi \to \psi(m) = \phi(m)\chi(\pi(m))^{-1}$ is a bijection of the gauge group $\tilde{M}^G$ onto the set of all equivariant connections.*

If the action of G is free, it is customary to consider the connection form ω as the form with values in the Lie algebra $\mathfrak{g}$. This form is defined by the left Maurer-Cartan form τ on G via

$$\omega_\phi = \phi^*\tau.$$

Recall that the curvature (Frobenius) form of the transversal connection $\mathcal{H}$ is the mapping $\Psi : \mathcal{H} \times \mathcal{H} \to \mathcal{F}$: $\Psi(X,Y)_m = P_m[X,Y]$, [11, Section 1.5]. The form $\Psi \circ h_{\mathcal{H}} \in \Lambda^2 T(M) \otimes \mathcal{F}$ is called the curvature form of the connection ω. Here $h_{\mathcal{H}}$ is the projection onto the horizontal subspace H_m at each point $m \in M$. The local integrability of the distribution H_m is equivalent to the nullity of the curvature form. In the case that the action of G is free and $\mathcal{F}$ is a G-orbit foliation, the curvature form Ω_ω of the connection ω take values in the Lie algebra $\mathfrak{g}$ and can be calculated by the structural equation, [8],

$$\Omega(m)(X,Y) = d\omega(m)(X,Y) + \frac{1}{2}[\omega(m)(X), \omega(m)(Y)], \quad m \in M,\ X,Y \in T_m(M).$$

Let ρ be a Riemannian metric on the manifold M. The metric ρ defines a corresponding transversal connection α where the horizontal subspace is the ρ-orthogonal complement of $V_m = T_m(\mathcal{F}_m)$. The vertical projection v_m of the connection α is orthogonal projection $P^\perp$ to the subspace $T_m(\mathcal{F})$. The connection α is called a *mechanical connection* in Hamiltonian mechanics, [12].

Now we return to the general case of a foliated Riemannian manifold $(M, \rho, \mathcal{F})$. The metric ρ defines an operator norm $\|\cdot\|$ on the tangent space at a point $m \in M$. Observe that the norm $\|P_m\|$ of a projection $P_m : T_m(M) \to V_m$ satisfies the inequality $\|P_m\| \geq 1$ and that the orthogonal projection $P_m^\perp$ is the unique operator with norm one at each point, [6]. As a result the mechanical connection is the unique pointwise minimizer of the functional

$$E_D(\omega) = \int_D \|P_m\|^2 dV_\rho,$$

defined on the space $Con_{\mathcal{F}}(M)$ for an arbitrary compact subset $D \subset M$. Here dV_ρ is the volume element defined by the metric ρ.

Definition 4. *Let $u \in \mathcal{X}(M)$ be a vector field on M. Let ω be a transversal connection with respect to a foliation $\mathcal{F}$ on M. Define the polarized energy of a connection ω with respect to u on the compact subset D via*

$$E_{(u,D)}(\omega) = \int_D \frac{\|v_\omega(m)(u_m)\|^2}{\|v_\omega(m)\|^2} dV_\rho. \tag{3.2}$$

Definition 5. *We call the critical points of the functional $E_{(u,D)}(\omega)$ in the space $Con_{\mathcal{F}}(M)$ harmonic connections polarized by the vector field u. Points of maximum of this functional will be called optimal harmonic connections.*

Remark 1. Points of *minimum* of the functional (3.2) are also of interest. In particular, a pointwise minimum of $E_{(u,D)}(\omega)$ satisfies $\omega(u(m)) = 0$ for all $m \in M$. Geometrically this means that the vector field u is ω-horizontal.

Consider the evolution of ω under the phase flow ϕ^t of the vector field u:

$$\begin{aligned}\dot{\omega} &= \phi^{t*}\omega \\ &= di_u\omega + i_u d\omega \\ &= i_u d\omega \\ &= d\omega(u, (h_\omega + v_\omega)(\cdot)) \\ &= \Omega(u, \cdot) + d\omega(u, v_\omega(\cdot)).\end{aligned}$$

Thus, the evolution of the connection ω under the phase flow ϕ^t is partly controlled by the curvature Ω of the connection ω.

Suppose now that the foliation is by orbits of a free and proper action of a Lie group G. If the connection ω is globally integrable and G-equivariant, i.e. given by an equivariant G-frame $\phi : M \to G$:

$$\omega(m) = T_m(\phi^{-1}(\phi(m)) = \phi^* \tau_L$$

for a surjective mapping $\phi : M \to G$ where τ_L is the left Maurer-Cartan form on G, then the curvature Ω vanishes.

Let us prove now that $d\omega(u, v_\omega(X)) = 0$ for all vectors $X \in T_m(M)$. We have

$$d\omega(u, \xi) = u \cdot \omega(\xi) - \xi \cdot \omega(u) - \omega([u, \xi]) \tag{3.3}$$

for $\xi = v_\omega(X)$ an arbitrary vector field tangent to G orbits. First note that the term $\xi \cdot \omega(u)$ vanishes since $\omega(u) = 0$. Now, the expression (3.3) is linear in ξ, so it is sufficient to prove it is zero for $\xi = \eta_M,\ \eta \in \mathfrak{g}$. For such ξ, $\omega(\xi) = \omega(\eta_M) = \eta$ due to the equivariance of ϕ, so the first term in (3.3) vanishes. Finally, since G is a symmetry of the vector field u, $[\eta_M, u] = -L_{\eta_M} u = 0$, so the last term in (3.3) vanishes as well. As a result we have proved

Proposition 1. *Let $\phi : M \to G$ be an equivariant G-frame that delivers a pointwise zero minimum to the polarized energy functional* (3.2). *Then the Maurer-Cartan connection $\omega_\phi = \phi^*(\tau_L)$ is invariant under the flow of the vector field u:*

$$\dot{\omega}_\phi = \mathcal{L}_u \omega = 0.$$

This demonstrates the relation of the polarized energy functional to the *integrals of motion* of the flow ϕ^t. This also motivates an interest in studying the maxima of the functional (3.2).

4. Pointwise optimal harmonic connections

As the starting point in the study of harmonic connections for a vector field u we consider the problem of pointwise maximization of the density of the functional (3.2). That is, we ask if there exists a subspace $H_m \subset T_m(M)$ complementary to V_m that maximizes the density function

$$H_m \to \frac{\|v_m^H(u(m))\|}{\|v_m^H\|}.$$

Here v_m^H is the projection in $T_m(M)$ onto V_m with kernel H_m and $\|v_m^H\|$ denotes the operator norm, in the space $\mathrm{Hom}(T_m(M), V_m)$, induced by the Riemannian metric. This is a problem of linear algebra. We begin with the following lemma.

Lemma 4. *Let: $P : E \to E$ be projection in a Euclidean vector space (E, ρ). Let (W, ρ) be an P-invariant subspace of E such that $Im(P) \subsetneqq W$. Denote by P_W the projection equal to P on W and zero on the orthogonal complement $W^\perp$ to W in E. Then $\|P_W\| \leqslant \|P\|$, where $\|\cdot\|$ is the operator norm induced by ρ.*

Proof. Let $u = u_W + u^\perp$ denote the direct sum decomposition of a vector $u \in E = W \oplus W^\perp$. Recall the definition $\|P\|^2 = \sup\limits_{u \in E, u \neq 0} \dfrac{\|Pu\|^2}{\|u\|^2}$. Then,

$$
\begin{aligned}
\|P_W\|^2 &= \sup_{u=u_W+u^\perp\in W\oplus W^\perp} \frac{\|P_W u\|^2}{\|u\|^2} \\
&= \sup_{u=u_W+u^\perp\in W\oplus W^\perp,\ u_W\neq 0} \frac{\|Pu_W\|^2}{\|u_W\|^2+\|u^\perp\|^2} \\
&\leqslant \sup_{u=u_W\in W} \frac{\|Pu_W\|^2}{\|u_W\|^2} \\
&\leqslant \sup_{u\in E, u\neq 0} \frac{\|Pu\|^2}{\|u\|^2} = \|P\|^2.
\end{aligned}
$$

□

Remark 2. The case $W = Im(P)$ is not excluded from the above Lemma.

Let ω be any connection. Denote by ω_W the connection constructed at each point $m \in M$ as in Lemma 4 with $W = W_m = V_m \oplus \mathbb{R}u(m)$. It is possible that $u(m) \in V_m$, in which case $W = V_m$. It is clear that $m \to \omega_W(m)$ depends smoothly on the point m and that the projection v_{ω_W} associated to the connection ω_W satisfies $v_{\omega_W,m}(u(m)) = v_{\omega,m}(u(m))$ at each point m.

Considering the polarized energy functional $E_{(u,D)}(\omega)$ we see that $E_{(u,D)}(\omega) \leqslant E_{(u,D)}(\omega_W)$, since pointwise we have

$$
\frac{\|v^\omega_m(u(m))\|^2}{\|v^\omega_m\|^2} \leq \frac{\|v^{\omega_W}_m(u(m))\|^2}{\|v^{\omega_W}_m\|^2}.
$$

Therefore to find the points of maximum of this functional we may restrict to the subset of connections $Con_{\mathcal{F}}(M)$ with the property $\omega|_{W^\perp} = 0$.

Remark 3. If, in addition, the metric ρ and the connection ω are invariant under an action of a Lie Group G, acting on M and preserving the foliation $\mathcal{F}$, then $g^t_* u \in \mathbb{R}u + T\mathcal{F}$ and it follows from this that the connection ω_W is G-invariant as well.

Our first goal is to calculate the denominator $\|v^{\omega_W}_m\|^2$ in the expression for the polarized energy density. As a reminder, the norm employed here is the operator norm defined by the norm generated by the metric ρ in $T_m(M)$. Let $u(m) = u^\perp(m) + u^0(m)$ be the orthogonal decomposition of the vector $u(m)$ in $T_m(M) = V_m \oplus V_m^\perp$.

In the following, we fix a point $m \in M$ and refrain from reference to m to simplify notation. Also, denote the projection $v^{\omega_W}_m$ by P. Assume that the vectors $u, u^\perp$ and $Pu^\perp$ are linearly independent and let K be the three-dimensional subspace of W spanned by these vectors. Note that since $Pu = Pu^\perp + u^0$, this is the same subspace as spanned by $u^0, u^\perp$ and Pu. Denote by V_0 the subspace of V orthogonal to the vectors $Pu^\perp$ and u^0. Then we have the orthogonal decomposition:

$$
T_m(M) = V_0 \oplus K \oplus W^\perp.
$$

Corresponding to this decomposition and choosing a basis in the linear span $span\{u^0, Pu^\perp\} \subseteq K$ accordingly, we get a representation of the projection P in block-diagonal form:

$$P = \begin{pmatrix} E_{k-2} & 0 & 0 \\ 0 & P_3 = \begin{pmatrix} 1 & 0 & \lambda_1 \\ 0 & 1 & \lambda_2 \\ 0 & 0 & 1 \end{pmatrix} & 0 \\ 0 & 0 & 0_{n-k-1} \end{pmatrix}. \tag{4.1}$$

Since ρ restricts to a symmetric bilinear form on V_0, K and $W^\perp$, orthogonal transformations may be made in each subspace to diagonalize ρ. Thus one may choose an orthonormal basis e_i in all three subspaces such that the metric ρ takes the form $\rho(u,v) = \sum_i c_i^2 v_i u_i$ for some positive coefficients c_i. In this new basis the matrix for P has the form

$$Q^{\mathrm{T}} P Q = \begin{pmatrix} E_{k-2} & 0 & 0 \\ 0 & Q_3^{\mathrm{T}} P_3 Q_3 & 0 \\ 0 & 0 & 0_{n-k-1} \end{pmatrix},$$

where Q and Q_3 give the change of basis that diagonalizes ρ in the spaces $T_m(M)$ and K, respectively.

In the basis $f_i = c_i e_i$ the scalar product ρ has the canonical Euclidean form $\rho(v,w) = \sum_i v_i w_i$ and the matrix for P becomes $P_1 = C^{-1} Q^{\mathrm{T}} P Q C$, where C is the diagonal matrix with diagonal entries c_i . Since ρ has Euclidean form in the basis f_i, the operator norm of the projection P may be computed as the spectral norm of the matrix P_1, that is, $\|P_1\|^2 = \max\{|\mu| \; \mu \text{ is an eigenvalue of } P_1^* P_1\}$. Because they are related by a similarity, the eigenvalues of $P_1^* P_1$ are the same as those of $P^* P$. The matrix $P^* P$ in the basis from (4.1) has the following block diagonal form:

$$P^* P = \begin{pmatrix} E_{k-2} & 0 & 0 \\ 0 & P_3^* P_3 & 0 \\ 0 & 0 & 0_{n-k-1} \end{pmatrix}$$

Thus it remains only to compute the eigenvalues of $P_3^* P_3$.

Let e_1,e_2 and e_3 be the orthonormal basis of K such that $u^0 = \|u^0\| e_1$, and $u^\perp = \|u^\perp\| e_3$. Let $Pe_3 = \xi e_1 + \eta e_2$, then $Pu^\perp = \|u^\perp\|(\xi e_1 + \eta e_2)$ and $Pu = u^0 + Pu^\perp$. In the basis e_i, the matrix of P_3 is

$$P_3 = \begin{pmatrix} 1 & 0 & \xi \\ 0 & 1 & \eta \\ 0 & 0 & 0 \end{pmatrix},$$

and

$$P_3^* P_3 = \begin{pmatrix} 1 & 0 & \xi \\ 0 & 1 & \eta \\ \xi & \eta & \xi^2 + \eta^2 \end{pmatrix}.$$

The eigenvalues of this matrix are $1, 0$, and $1 + \xi^2 + \eta^2$. As a result, the square of the norm of the projection P is $1 + \xi^2 + \eta^2$. On the other hand, $\xi^2 + \eta^2 = \frac{\|Pu^\perp\|_\rho^2}{\|u^\perp\|_\rho^2}$.

Now we return to the assumption that the vectors $u, u^\perp, Pu^\perp$ are linearly independent. If they are not linearly independent there are two possibilities: $u^0 = 0$ or $Pu^\perp = \lambda u^0$.

Assume first that $u^0 = 0$. Choose a basis $\{e_1, e_2\}$ for K (now two dimensional) such that $u^\perp = \|u^\perp\| e_2$ and write $Pe_2 = \xi e_1$. The matrix for the projection restricted to K is now

$$P_2 = \begin{pmatrix} 1 & \xi \\ 0 & 0 \end{pmatrix}.$$

Thus

$$P_2^* P_2 = \begin{pmatrix} 1 & \xi \\ \xi & \xi^2 \end{pmatrix}$$

has eigenvalues 0 and $1 + \xi^2$. On the other hand, $\xi^2 = \frac{\|Pu^\perp\|^2}{\|u^\perp\|^2}$.

Now assume that $Pu^\perp = \lambda u^0$. Choose a basis $\{e_1, e_2\}$ for K such that $u^0 = \|u^0\| e_1$ and $u^\perp = \|u^\perp\| e_2$. Write $Pe_2 = \xi e_1$. The matrix for the projection restricted to K is again

$$P_2 = \begin{pmatrix} 1 & \xi \\ 0 & 0 \end{pmatrix}.$$

Noting that $\xi^2 = \frac{\|Pu^\perp\|^2}{\|u^\perp\|^2}$ we arrive at the same expression as the previous cases.

We have proved the following lemma.

Lemma 5. *Let m, u, and W be as in the above discussion. Then*

$$\|v_m^{\omega_W}\|^2 = 1 + \frac{\|v_m^{\omega_W} u^\perp\|^2}{\|u^\perp\|^2}$$

We now use the expression from this lemma to calculate the point-wise supremum of the energy functional. The result is the following:

Proposition 2. *Let $u = u^0 + u^\perp$. Assume that $||u^\perp|| \neq 0$. Then*

$$\sup_\omega \frac{\|v_m^\omega u\|^2}{\|v_m^\omega\|^2} = \|u^0\|^2 + \|u^\perp\|^2$$

If $u^0 = 0$, this supremum is not achieved by any choice of connection ω, but approached as $\|v^\omega u^\perp\| \to \infty$. If $u^0 \neq 0$, there is a unique connection that gives the supremum.

Proof. Without loss of generality we restrict to connections ω with vertical projections v_m^ω non-zero only on the subspace $W = V \oplus \mathbb{R}u$. Again, reference to the point $m \in M$ is dropped. By Lemma 5,

$$\|v^\omega\|^2 = 1 + \frac{\|v^\omega u^\perp\|^2}{\|u^\perp\|^2}.$$

Thus we wish to find the supremum over all choices of connection ω of the expression

$$\frac{\|v^\omega u\|^2}{1 + \frac{\|v^\omega u^\perp\|^2}{\|u^\perp\|^2}} = \frac{\|u^\perp\|^2 \|v^\omega u\|^2}{\|u^\perp\|^2 + \|v^\omega u^\perp\|^2}.$$

Since $u^0 \in V$, $v^\omega(u) = u^0 + v^\omega u^\perp$. Write $v^\omega u^\perp = w + \lambda u^0$ where $\lambda \in \mathbb{R}$ and $\rho(w, u^0) = 0$. Then

$$\begin{aligned}\frac{\|u^\perp\|^2 \|v^\omega u\|^2}{\|u^\perp\|^2 + \|v^\omega u^\perp\|^2} &= \frac{\|u^\perp\|^2 \|u^0 + v^\omega u^\perp\|^2}{\|u^\perp\|^2 + \|v^\omega u^\perp\|^2} \\ &= \frac{\|u^\perp\|^2 \|u^0 + w + \lambda u^0\|^2}{\|u^\perp\|^2 + \|w + \lambda u^0\|^2} \\ &= \frac{\|u^\perp\|^2 \Big((\lambda+1)^2 \|u^0\|^2 + \|w\|^2 \Big)}{\|u^\perp\|^2 + \|w\|^2 + \lambda \|u^0\|^2}.\end{aligned}$$

Clearly, λ and w determine the choice of ω. Employ the notation $\mu^2 = \|w\|^2$, $A = \|u^\perp\|^2$ and $B = \|u^0\|^2$. We are then searching for the following supremum

$$\sup_{\lambda,\mu} \frac{(\lambda+1)^2 AB + \mu^2 A}{A + \mu^2 + \lambda^2 B} \tag{4.2}$$

where $A > 0$ and $B \geq 0$. If $B = \|u^0\|^2 = 0$, that is $u^0 = 0$, the expression becomes

$$\sup_\mu \frac{A\mu^2}{A + \mu^2}.$$

It is simple to verify for this expression that the supremum is $A = \|u^\perp\|^2$, approached as $\mu \to \infty$. We may thus assume that $B \neq 0$.

Returning to (4.2), computing the partial derivatives with respect to μ and λ we arrive at the following conditions for critical points:

$$2(\lambda+1)AB\big(A + \mu^2 + \lambda^2 B\big) - \big((\lambda+1)^2 AB + \mu^2 A\big) 2\lambda B = 0, \tag{4.3a}$$

$$2\mu A\big(A + \mu^2 + \lambda^2 B\big) - \big((\lambda+1)^2 AB + \mu^2 A\big) 2\mu = 0. \tag{4.3b}$$

If $\mu \neq 0$, we may conclude from (4.3b) that

$$A\big(A + \mu^2 + \lambda^2 B\big) = \big((\lambda+1)^2 AB + \mu^2 A\big),$$

which, being substituted into (4.3a), allows us to conclude, since $A \neq 0$ and $\big(A + \mu^2 + \lambda^2 B\big) \neq 0$, that

$$2(\lambda+1)B = 2\lambda B.$$

Since we have assumed $B \neq 0$, this equality is clearly impossible for any λ to satisfy. We are forced to conclude that if $B \neq 0$, $\mu = 0$.

Assuming that $\mu = 0$, the expression (4.2) becomes

$$\sup_\lambda \frac{(\lambda+1)^2 AB}{A + \lambda^2 B}. \tag{4.4}$$

Notice that as $\lambda \to \infty$, this expression tends to A. The condition for critical points is

$$2(\lambda+1)AB(A + \lambda^2 B) - 2\lambda B(\lambda+1)^2 AB = 0,$$

or more simply

$$(\lambda+1)(A + \lambda^2 B) - \lambda B(\lambda+1)^2 = 0.$$

If $\lambda + 1 = 0$, the expression (4.4) is 0, which cannot possibly be the supremum, so we assume $\lambda + 1 \neq 0$ and arrive at the condition

$$A + \lambda^2 B = \lambda(\lambda + 1)B,$$

that is $\lambda = A/B$. Put this into expression (4.4) to conclude that

$$\sup_{\lambda} \frac{(\lambda+1)^2 AB}{A + \lambda^2 B} = A + B,$$

and thus finally that

$$\sup_{\lambda,\mu} \frac{(\lambda+1)^2 AB + \mu^2 A}{A + \mu^2 + \lambda^2 B} = A + B.$$

$\square$

Corollary 1. *Let $u = u^0 + u^\perp$ as above and assume $u^0 \neq 0$. Let ω be the connection giving the maximum calculated above. Then*

$$v^\omega(u) = \left(1 + \frac{\|u^\perp\|^2}{\|u^0\|^2}\right) u^0$$

Proof. A calculation:

$$v^\omega(u) = v^\omega(u^0 + u^\perp) = u^0 + v + \lambda u^0,$$

but we concluded that $\mu = 0$ (so $v = 0$) and that $\lambda = \frac{\|u^\perp\|^2}{\|u^0\|^2}$, so

$$u^0 + v + \lambda u^0 = \left(1 + \frac{\|u^\perp\|^2}{\|u^0\|^2}\right) u^0.$$

$\square$

We now write down an expression for the maximal connection found above. In a local frame $\{X_i\}$ of the subbundle $T\mathcal{F} \subset T(M)$, we may specify a connection ω by its vertical projection v^ω, which may be written

$$v^\omega = \sum_{i=1}^{k} X_i \otimes \omega_i,$$

where $\omega_i \in \Omega^1(U)$ (U a neighborhood of m) are scalar 1-forms. At the same time each form ω_i can be represented as

$$\omega_i(X) = \rho(X, Z_i)$$

where the vector fields Z_i are uniquely defined on U. If $\omega \in Con_{\mathcal{F}}(M)$ then $Z_i(m) \in W_m$ for all m. The following proposition gives an expression for the Z_i in the case of the maximal connection.

Proposition 3. *Assume that $u^0 \neq 0$. Let X_i be a basis for $V \subseteq T_m(M)$ and Y_i a basis for V dual to X_i with respect to ρ. The vertical projection for the maximal connection found above is given by*

$$v^\omega(\cdot) = \sum_i \rho(\cdot, Z_i) \otimes X_i,$$

where

$$Z_i = Y_i + \frac{u^{oi}}{\|u^0\|^2} u^\perp.$$

Proof. Let $w \in V \oplus \mathbb{R}u$ and write

$$w = \sum_i \alpha_i X_i + \lambda u.$$

Then

$$v^\omega(w) = \sum_i \alpha_i X_i + \lambda\left(1 + \frac{\|u^\perp\|^2}{\|u^0\|^2}\right)u^0.$$

Now, $u^0 = \sum_i \rho(u, Y_i)X_i$, so

$$v^\omega(w) = \sum_i \left[\alpha_i + \lambda\rho(u, Y_i)\left(1 + \frac{\|u^\perp\|^2}{\|u^0\|^2}\right)\right]X_i.$$

Also, $w - \lambda u = \sum_i \alpha_i X_i$, so

$$\alpha_i = \rho(w - \lambda u, Y_i) = \rho(w, Y_i) - \lambda\rho(u, Y_i),$$

and thus

$$\begin{aligned} v^\omega(w) &= \sum_i \left[\rho(w, Y_i) - \lambda\rho(u, Y_i) + \lambda\rho(u, Y_i)\left(1 + \frac{\|u^\perp\|^2}{\|u^0\|^2}\right)\right]X_i \\ &= \sum_i \left[\rho(w, Y_i) + \lambda\rho(u, Y_i)\left(\frac{\|u^\perp\|^2}{\|u^0\|^2}\right)\right]X_i. \end{aligned}$$

But, since $\rho(w, u^\perp) = \lambda\|u^\perp\|^2$,

$$\begin{aligned} v^\omega(w) &= \sum_i \left[\rho(w, Y_i) + \rho(u, Y_i)\frac{1}{\|u^0\|^2}\rho(w, u^\perp)\right]X_i \\ &= \sum_i \rho\left(w, Y_i + \rho(u, Y_i)\frac{1}{\|u^0\|^2}u^\perp\right)X_i. \end{aligned}$$

What is left is to notice that $\rho(u, Y_i) = \rho(u^0, Y_i) = u^{0i}$. $\square$

Remark 4. We introduced above a local basis of the vector fields $X_i,\ i = 1, \dots k$ tangent to the foliation $\mathcal{F}$ and the local basis Y_i of vector fields on the same foliation dual to the basis X_i via the metric ρ: $\rho(X_i, Y_j) = \delta_{ij}$. We could here use an orthonormal basis of V_m. But later we will pick $X_i = \xi_{iM}$ where ξ_i is some basis of the Lie algebra $\mathfrak{g}$ under the action mapping $\mathfrak{g} \to T_m M$, and this basis is not necessary orthonormal.

We summarize the results of this section in the following theorem.

Theorem 1. *Let (M, ρ) be a Riemannian manifold of dimension n and $\mathcal{F}$ a foliation of dimension k in M. Let X_i be a basis of $V_m = T_m\mathcal{F}$ and Y_i a basis of V_m dual to X_i with respect to ρ, so that $\rho(X_i, Y_j) = \delta_{ij}$.*

Let $u(m) = u^0(m) + u^\perp(m)$ be the orthogonal decomposition of a vector $u(m) \in T_m(M) = V_m \oplus V_m^\perp$ and assume $u^\perp(m) \neq 0$. For any point $m \in M$ one of the following two alternatives is true:

(1) $u^0(m) = 0$. *The supremum of the polarized energy density is $\|u^\perp(m)\|^2$ and is approached as $\|v_m^\omega u^\perp\| \to \infty$.*

(2) $u^0(m) \neq 0$. *The supremum of the polarized energy density is* $\|u(m)\|^2 = \|u^\perp(m)\|^2 + \|u^0(m)\|^2$, *achieved at a unique maximal connection* ω_u *whose vertical projection is given by*

$$v_m^{\omega_u}(\cdot) = \sum_i \rho(\cdot, Z_i(m)) \otimes X_i(m) \tag{4.5}$$

where

$$Z_i(m) = Y_i(m) + \frac{u^{0i}}{\|u^0\|^2} u^\perp(m). \tag{4.6}$$

Corollary 2. *The horizontal subspace of the connection* ω_u *at a point* $m \in M$ *is the linear span of the subspace* $W^\perp$ *orthogonal to the subspace* $W = \langle V, u\rangle$ *and the vector*

$$u^\perp - \frac{\|u^\perp(m)\|^2}{\|u^0(m)\|^2} u^{0k} X_k = u^\perp - \frac{\|u^\perp(m)\|^2}{\|u^0(m)\|^2} u^0.$$

Proof. Let $\xi = \xi^i X_i + \xi^u u^\perp + \xi^j w_j \in T_m(M)$ be a tangent vector, here w_i is a basis of $W^\perp$. Then,

$$v^{\omega_u}(m)(\xi) = \sum_k \left(\xi^k + \frac{\|u^\perp(m)\|^2}{\|u^0(m)\|^2} u^{0k} \xi^u \right) X_k.$$

Equating to zero each component of last sum we express the coefficients ξ^k through ξ^u: $\xi^k = \frac{\|u^\perp(m)\|^2}{\|u^0(m)\|^2} u^{0k}\xi^u$. The conclusion follows immediately. □

Remark 5. The maximal connection ω_u depends on the direction of the vector field u but not on its magnitude; the expressions (4.6) for the vector fields Z_i are invariant with respect to the multiplication of u by any smooth non-zero function $f(m)$: $u(m) \to f(m)u(m)$. Thus, the connection ω_u is defined by the *direction field*, or *linear subbundle* $\mathbb{R}u$ of the tangent bundle $T(X)$ rather than the vector field u itself.

Remark 6. Generically, for a vector field u, $u^0(m) \neq 0$. As a result, generically, the maximal connection ω_u exists, is unique, and may have singularities at the points where $u^0 = 0$.

5. Connections on a principal bundle

In this section we discuss a special case of Section 3 where the foliation $\mathcal{F}$ is formed by the orbits of a free action of a compact Lie group G preserving the metric ρ. The manifold M is, in this case, a principal G-bundle over the manifold $M' = M/G$. In this case there is a canonically defined connection α with horizontal space at each point orthogonal to the tangent space of a G-orbit: $H_m = \mathfrak{g}_m^\perp$. Although this connection, known as the *mechanical connection*, looks trivial, its relation to other structures present, e.g. the momentum map J of the lift of the G-action to the cotangent bundle $T^*(M)$, the Legendre transformation $\mathcal{L} : T(M) \to T^*(M)$, etc., leads to beautiful and useful results, [4, 5, 13].

We begin by identifying the connection form α of the mechanical connection with the $\mathfrak{g}$-valued 1-form α on M and give a formula for $\omega_u(m)$ in terms of α and the one-form $u^\flat$ corresponding to the vector field u. After that we calculate the curvature of this maximal connection ω_u.

First notice that if G acts by isometries of the metric ρ then the connection α is G-equivariant. The same is true for the maximal connection ω_u. To see this we start with the simple case of a free action of a connected abelian Lie group.

Lemma 6. *Let G be a connected abelian Lie group acting by isometries of the metric ρ and let u be a G-invariant vector field: $g_* u = u$. Then the maximal connection ω_u is G-invariant.*

Proof. For $m \in M$, $g \in G$ and $\xi \in \mathfrak{g}$ we have

$$g\ \exp(t\xi)m = \exp(Ad_g \xi)m.$$

Taking the derivative by t at $t = 0$ we get

$$g_{*m}\xi_M(m) = Ad_g(\xi)_M(gm).$$

In the abelian case this proves that the vector field ξ_M is equivariant. Choosing a basis ξ_i of the Lie algebra $\mathfrak{g}$ we may take $X_i = \xi_{iM}$ in the formulation of Theorem 1 to ensure that vector fields X_i are G-invariant. Since G acts by isometries, the Y_i are G-invariant as well. The decomposition $u = u^0 + u^\perp$ is also G-invariant. As a result, all terms in the formula for the connection form ω_u are G-invariant, so the form is G-invariant. $\square$

Proposition 4. *Suppose the foliation $\mathcal{F}$ is defined by an action of a Lie group G with all orbits having the same dimension. Then, the maximal connection ω_u is G-invariant.*

Proof. Let $g \in G,\ m \in M$. Since u is G-invariant, $g_{*m}u(m) = u(gm)$. Since g preserves the orbit Gm and acts by isometries of metric ρ, it preserves the orthogonal decomposition $u = u^0 + u^\perp$:

$$g_{*m}u^0(m) = u^0_{gm}, \quad \text{and} \quad g_{*m}u^\perp(m) = u^\perp(gm).$$

In particular, in the formula (4.6), $\|u^0(gm)\|^2 = \|u^0(m)\|^2$.

On the other hand, if $\{X_i(y)\}$ is an orthonormal frame of $\mathfrak{g}_y$ in a neighborhood of the point m, then $\{g_{*y}X_i(y)\}$ is an orthonormal frame of $\mathfrak{g}_{gy}$ in a neighborhood of the point gm. It is clear that the vector fields $g_{*y}Y_i(y)$ are dual to $\{g_{*y}X_i(y)\}$ basis of the tangent space to the orbit $\mathfrak{g}_{gy}$ in the same neighborhood of gm. Writing the projection $v^{\omega_u}(gm)$ in the basis $\{g_{*y}X_i(y)\}$ we get

$$v^{\omega_u}(gm)(\cdot) = \sum_i \rho(\cdot, Z_i(gm)) \otimes g_{*m} X_i(m),$$

where

$$\begin{aligned}
Z_i(gm) &= g_{*m} Y_i(m) + \frac{1}{\|u^0(gm)\|^2} \rho(g_{*m} Y_i(m), u(gm)) u^{\perp}(gm) \\
&= g_{*m} Y_i(m) + \frac{1}{\|u^0(m)\|^2} \rho(g_{*m} Y_i(m), g_{*m} u(m)) g_{*m} u^{\perp}(m) \\
&= g_{*m} \left(Y_i(m) + \frac{1}{\|u^0(m)\|^2} \rho(Y_i(m), u(m)) u^{\perp}(m) \right) \\
&= g_{*m} Z_i(m).
\end{aligned}$$

As a result,

$$\begin{aligned}
v^{\omega_u}(gm)(\xi) &= \sum_i \rho(\xi, g_{*m} Z_i(m)) \otimes g_{*m} X_i(m) \\
&= g_{*m} \left(\sum_i \rho(g_{*m}^{-1} \xi, Z_i(m)) \otimes X_i(m) \right) \\
&= g_{*m} \left(v^{\omega_u}(m)(g_{*m}^{-1} \xi) \right) \\
&= \left(g_{*m} v^{\omega_u}(m) \right)(\xi).
\end{aligned}$$

□

Let ξ_i be a basis of the Lie algebra $\mathfrak{g}$. As before, choose a basis X_i of the subspace $\mathfrak{g}_m$ of the form $X_i(m) = \xi_{iM}(m)$. Recall the notation $\flat : T(M) \to T^*(M)$ for index lowering induced by the metric ρ.

Recall that the vertical projection of the connection α is the orthogonal projection to the tangent space to G-orbits: $v_\alpha = P^{\perp}$, and horizontal projection is the orthogonal projection: $h_\alpha = I - P^{\perp}$. Therefore,

$$\alpha(-) = \sum_i \rho(Y_i, -) \otimes \xi_{iM}.$$

Combining this expression with the formulas (4.5), (4.6) we arrive at the $\mathfrak{g}$-valued connection form

$$\begin{aligned}
\omega_u(m)(-) &= \alpha(-) + \frac{1}{\|u^0\|^2} \alpha(u) \otimes u^{\perp\,\flat}(-) \\
&= \alpha(-) + \frac{u^{0i}}{\|u^0\|^2} u^{\perp\,\flat}(-) \otimes \xi_{iM},
\end{aligned}$$

Note that $\omega_u = \alpha$ at the points where $u^{\perp} = 0$.

Corollary 3. *If $u(m) \in T_m(Gm)$ (i.e. $u(m)$ is tangent to G-orbits), then*

$$\omega_u = \alpha.$$

To get an expression for the curvature $\Omega_u(m)$ of the maximal connection ω_u in terms of the curvature Ω^α of the mechanical connection α we employ the structural equation [8],

$$\begin{aligned}\Omega_u(m)(X,Y) &= d\omega_u(m)(X,Y) + [\omega_u(m)(X), \omega_u(m)(Y)]\\ &= d\alpha(X,Y) + d\left(\frac{u^{0i}}{\|u^0\|^2}u^{\perp\flat}\otimes\xi_i\right)(X,Y)\\ &\quad + \left[\alpha(X) + \frac{u^{0i}}{\|u^0\|^2}u^{\perp\flat}(X)\xi_i, \alpha(Y) + \frac{u^{0j}}{\|u^0\|^2}u^{\perp\flat}(Y)\xi_j\right]\\ &= \Omega^\alpha(X,Y) + d\left(\frac{u^{0i}}{\|u^0\|^2}u^{\perp\flat}(-)\right)(X,Y)\xi_i + [\alpha(X),\xi_j]\frac{u^{0j}}{\|u^0\|^2}u^{\perp\flat}(Y)\\ &\quad + [\xi_i, \alpha(Y)]\frac{u^{0i}}{\|u^0\|^2}u^{\perp\flat}(X) + \frac{u^{0i}}{\|u^0\|^2}\frac{u^{0j}}{\|u^0\|^2}u^{\perp\flat}(X)u^{\perp\flat}(Y)[\xi_i,\xi_j].\end{aligned} \tag{5.1}$$

Let $X^k = \rho(X, Y_k)$ and $Y^k = \rho(Y, Y_k)$. We simplify the last three terms in (5.1) as follows:

$$\begin{aligned}&[\alpha(X),\xi_j]\frac{u^{0j}}{\|u^0\|^2}u^{\perp\flat}(Y) = \frac{u^{0j}}{\|u^0\|^2}\rho(u^\perp, Y)X^k c^l_{kj}\xi_l,\\ &[\xi_i,\alpha(Y)]\frac{u^{0i}}{\|u^0\|^2}u^{\perp\flat}(X) = -\frac{u^{0i}}{\|u^0\|^2}\rho(u^\perp, X)Y^k c^l_{ki}\xi_l,\\ &\frac{u^{0i}}{\|u^0\|^2}\frac{u^{0j}}{\|u^0\|^2}u^{\perp\flat}(X)u^{\perp\flat}(Y)[\xi_i,\xi_j] = 0,\end{aligned}$$

where c^k_{ij} are the structural constants of $\mathfrak{g}$, and we have used the identities

$$\begin{gathered}u^{\perp\flat}(X) = \rho(u^\perp, X), \quad [\xi_i,\xi_j] = c^k_{ij}\xi_k, \quad \alpha(X) = X^k\xi_k, \quad \alpha(Y) = Y^k\xi_k,\\ [\alpha(X),\xi_j] = X^k c^l_{kj}\xi_l, \quad [\xi_i,\alpha(Y)] = Y^k c^l_{ik}\xi_l.\end{gathered}$$

The final simplification follows from the antisymmetry of the bracket $[\xi_i, \xi_j]$. For the second term after the last equality in (5.1) we have

$$\begin{aligned}d\left(\frac{u^{0i}}{\|u^0\|^2}u^{\perp\flat}(-)\right)(X,Y)\xi_i = \Bigg[&X\cdot\left(\frac{u^{0i}}{\|u^0\|^2}\rho(u^\perp, Y)\right)\\ &- Y\cdot\left(\frac{u^{0i}}{\|u^0\|^2}\rho(u^\perp, X)\right) - \frac{u^{0i}}{\|u^0\|^2}\rho(u^\perp, [X,Y])\Bigg]\xi_i.\end{aligned}$$

Collecting these simplifications together we get an expression for the curvature of the maximal connection:

$$\begin{aligned}\Omega_u(m)(X,Y) &= \Omega_\alpha(m)(X,Y)\\ &+ \left[X\cdot\left(\frac{u^{0i}}{\|u^0\|^2}\rho(u^\perp, Y)\right) - Y\cdot\left(\frac{u^{0i}}{\|u^0\|^2}\rho(u^\perp, X)\right) - \frac{u^{0i}}{\|u^0\|^2}\rho(u^\perp, [X,Y])\right]\xi_i\\ &+ \left[\frac{u^{0j}}{\|u^0\|^2}\rho(u^\perp, Y)X^k c^l_{kj} - \frac{u^{0i}}{\|u^0\|^2}\rho(u^\perp, X)Y^k c^l_{ki}\right]\xi_l.\end{aligned}$$

Note that for an abelian Lie group G last two terms in this expression vanish.

Recall that the vanishing of the curvature is a necessary and locally sufficient condition for integrability of a connection. As a result we get

Proposition 5. *Let a foliation $\mathcal{F}$ be defined by the action of a Lie group G i.e. $\mathcal{F}_m = Gm$ by an action with one type of orbits G/K. Let α be the mechanical connection in (M, ρ). The maximal connection ω_u is locally integrable if and only if*

$$\Omega_u = \Omega_\alpha + d(\|u^0\|^{-2} u^0 \otimes u^{\perp\,\flat}) + \|u^0\|^{-2}[\alpha(-), u^0] \wedge u^{\perp\,\flat} = 0. \tag{5.2}$$

To compare the curvatures Ω_α and Ω_u of the mechanical and maximal connections, respectively, it is sufficient to consider the cases where arguments of the curvature form are either pairs of ω_u-horizontal (local) vector fields $X, Y \in W^\perp$ or pairs $X \in W^\perp, u_H = u^\perp - \frac{\|u^\perp(m)\|^2}{\|u^0(m)\|^2} u^{0k} X_k$. In fact, using formula (5.2) one can consider $(X \in W^\perp, u^\perp)$ instead of the last pair. We summarize these cases in the following Corollary.

Corollary 4.

(1) *For a pair (X, Y) with $X, Y \in W^\perp$,*

$$\Omega_{\omega_u}(m)(X, Y) = \Omega_\alpha(m)(X, Y) - \frac{u^{0i}}{\|u^0\|^2} \rho(u^\perp, [X, Y]) \xi_i.$$

(2) *For a pair (X, u_H) with $X \in W^\perp$,*

$$\begin{aligned} \Omega_{\omega_u}(m)(X, u_H) &= \Omega_\alpha(m)(X, u_H) \\ &\quad + \left[X \cdot \left(\frac{u^{0i}}{\|u^0\|^2} \|u^\perp\|^2 \right) - \frac{u^{0i}}{\|u^0\|^2} \rho(u^\perp, [X, u_H]) \right] \xi_i. \end{aligned}$$

(3) *The connection ω_u is flat if and only if, for all $X, Y \in W^\perp$,*

$$\Omega_\alpha(m)(X, Y) = \|u^0\|^{-2} \rho(u^\perp, [X, Y]) u^{0i} \xi_i, \tag{5.3}$$

and for all $X \in W^\perp$

$$\Omega_\alpha(m)(X, u_H) = -\left[X \cdot \left(\frac{u^{0i}}{\|u^0\|^2} \|u^\perp\|^2 \right) - \frac{u^{0i}}{\|u^0\|^2} \rho(u^\perp, [X, u_H]) \right] \xi_i. \tag{5.4}$$

Proof. The first statement follows directly from (5.2) where we use $\rho(u^\perp, X) = \rho(u^\perp, Y) = 0$. For the second statement we use in (5.2) $\rho(u^\perp, X) = 0$ and the fact that, for $X \in W^\perp$, the coefficients $X^k = \rho(X, Y_k)$ are zero. We also use the formula $\rho(u^\perp, u_H) = \|u^\perp\|^2$. The third statement follows directly from the first two. ☐

Example 1. Consider the case where there exists a G-frame $\phi : M \to G$ and the orthogonal complement $V_m^\perp = T_m(Gm)^\perp$ with respect to the metric ρ is integrable. The mechanical connection α is then flat: $\Omega_\alpha = 0$. The conditions for flatness of the connection ω_u take the form

$$\rho(u^\perp, [X, Y]) u^0 = 0, \qquad X, Y \in W^\perp \tag{5.5}$$

and

$$\left[X \cdot \left(\frac{\|u^\perp\|^2}{\|u^0\|^2} u^{0i} \right) + \frac{u^{0i}}{\|u^0\|^2} \rho(u^\perp, [X, u^\perp]) \right] = 0, \quad X \in W^\perp. \tag{5.6}$$

The component u^0 is generically nonzero. Therefore, the condition (5.5) generically takes the form

$$\rho(u^\perp, [X,Y]) = 0, \quad X, Y \in W^\perp,$$

In other words, the distribution $W^\perp$ is integrable.

Example 2. Suppose that G is one-dimensional. Let $\xi \in \mathfrak{g}$ and ξ_M the corresponding vector field on M. Denote by $\check{\xi}_M$ the dual basis of the distribution V, i.e. the vector field defined by $\rho(\check{\xi}_M, \xi_M) = 1$. We consider the distribution $\mathcal{W}$ generated by the vector fields u, ξ_M. This distribution is 2-dimensional at points where $u^\perp \neq 0$. As before, let $u = u^0 + u^\perp$ be the orthogonal decomposition of the vector field u. Introduce another vector $\tilde{u} = u^0 - u^\perp$. The vectors $u, \tilde{u}$ form a basis for $\mathcal{W}$ at the (generic) points where $u^\perp \neq 0$ and $u^0 \neq 0$. Denote by M_{gen} this open subset of points where both $u^\perp \neq 0$ and $u^0 \neq 0$.

Let $\mathbb{C}$ be the complex plane with coordinate $z = s+it$ and recall the complex derivatives $\partial_z = \frac{1}{2}(\partial_s - i\partial_t)$, $\partial_{\bar{z}} = \frac{1}{2}(\partial_s + i\partial_y)$. Choose a point $m_0 \in M$ and let $\gamma : \mathbb{C} \to M$ be given by

$$\gamma(s+it) = \exp(s \cdot u^0)\exp(t \cdot u^\perp) m_0,$$

where $\exp(s \cdot u^0)$ and $\exp(t \cdot u^\perp)$ are one-parameter groups of the phase flows of vector fields u^0 and $u^\perp$ respectively. This mapping defines a foliation $\mathcal{C}$ by 2-dimensional surfaces in M_{gen}; each of these surfaces being endowed with the complex structure inherited from $\mathbb{C}$. The distribution of tangent bundles to the leaves of this foliation coincide with the distribution $\mathcal{W}$: $T(\mathcal{C}_m) = \mathcal{W}_m$. In the leaves of this foliation the tangent mapping γ_* acts as

$$\gamma_*(\partial_s) = u^0, \quad \gamma_*(\partial_t) = u^\perp, \quad \gamma_*(\partial_z) = u, \quad \gamma_*(\partial_{\bar{z}}) = \tilde{u}.$$

As a result, u and $\tilde{u}$ generate holomorphic and anti-holomorphic sub-distributions of W in the decomposition

$$T(\mathcal{C}) = \mathrm{Hol}_{\mathcal{C}} \oplus \mathrm{AHol}_{\mathcal{C}}.$$

We now return to the conditions (5.3), (5.4) for the flatness of the connection ω_u. Recall that for the mechanical connection α, its V-valued curvature form is defined by

$$\Omega_\alpha(X,Y) = \mathrm{proj}_V\left([h_\alpha X, h_\alpha Y]\right)$$

for vector fields $X, Y \in \mathcal{X}(M)$. Here $\mathrm{proj}_{\mathcal{V}}$ is the ρ-orthogonal projection to the distribution V and $h_\alpha = I - \mathrm{proj}_V$ is the α-horizontal projection to $V^\perp$. In our one dimensional case, $\mathrm{proj}_V X = \rho(X, \check{\xi}_M)\xi_M$. Thus,

$$\Omega_\alpha(m)(X,Y) = \rho([h_\alpha X, h_\alpha Y], \check{\xi}_M)\xi_M, \quad X, Y \in \mathcal{X}(M). \tag{5.7}$$

Using this and the facts that $h_\alpha u_H = u^\perp$ and $h_\alpha X = X$ for all $X \in W^\perp$, and omitting the basis vector ξ_M, we rewrite conditions (5.3), (5.4) as

$$\rho([X,Y], \check{\xi}_M) = \|\bar{u}^0\|^{-2}\rho(u^\perp, [X,Y])u^0, \quad X, Y \in W^\perp \tag{5.8}$$

and

$$\rho([X, u^\perp], \check{\xi}_M) = -\left[X \cdot \left(\frac{u^0}{\|\bar{u}^0\|^2}\|u^\perp\|^2\right) - \frac{u^0}{\|u^0\|^2}\rho(u^\perp, [X, u_H])\right], \quad X \in W^\perp.$$

Here $\bar{u}^0 = u^0\xi_M$. Using $\|\bar{u}^0\|^2 = (u^0)^2\|\xi_M\|^2$ and $\check{\xi}_M = \|\xi_M\|^{-2}\xi_M$ in (5.7) we find that

$$\rho\left(\check{\xi}_M - \frac{1}{u^0\|\xi_M\|^2}u^\perp, [X,Y]\right) = 0, \quad X, Y \in W^\perp$$

if and only if

$$\rho(u^0 - u^\perp, [X,Y]) = 0, \quad X, Y \in W^\perp.$$

To transform the second condition (5.8) we notice that

$$\begin{aligned}[X, u_H] &= [X, u^\perp - \frac{u^0\|u^\perp\|^2}{\|\bar{u}^0\|^2}\xi_M] \\ &= [X, u^\perp] - X\cdot\left(\frac{u^0\|u^\perp\|^2}{\|\bar{u}^0\|^2}\right)\xi_M + \frac{u^0\|u^\perp\|^2}{\|\bar{u}^0\|^2}[X, \xi_M].\end{aligned}$$

Since G acts by isometries and leaves the distribution W invariant, it leaves invariant the orthogonal complement $W^\perp$. Therefore $[X, \xi_M] \in W^\perp$ and thus $\rho(u^\perp, [X, \xi_M]) = 0$. Additionally, $\rho(u^\perp, \xi_M) = 0$. Therefore,

$$\frac{u^0}{\|u^0\|^2}\rho(u^\perp, [X, u_H]) = \frac{1}{u^0\|\xi_M\|^2}\rho(u^\perp, [X, u^\perp]).$$

We use this to rewrite the condition (5.8) in the form

$$\rho\left(\check{\xi}_M - \frac{1}{u^0\|\xi_M\|^2}u^\perp, [X, u^\perp]\right) = -X\cdot\left(\frac{\|u^\perp\|^2}{u^0\|\xi_M\|^2}\right), \quad X \in W^\perp,$$

or

$$\rho(\bar{u}^0 - u^\perp, [X, u^\perp]) = -u^0\|\xi_M\|^2 X\cdot\left(\frac{\|u^\perp\|^2}{u^0\|\xi_M\|^2}\right), \quad X \in W^\perp,$$

Notice now that the connection ω_u depends on the direction of the vector field u but not on its magnitude. Thus, scaling the vector field u, i.e. multiplying both u^0 and $u^\perp$ by the same nonzero function $f(m)$, we may fulfill the condition

$$\frac{\|u^\perp\|^2}{u^0\|\xi_M\|^2} = 1, \quad \text{or} \quad u^0 = \frac{\|u^\perp\|^2}{\|\xi_M\|^2}. \tag{5.9}$$

We assume for the rest of this example (5.9) is satisfied. The second condition (5.8) of flatness then takes the form

$$\rho(\bar{u}^0 - u^\perp, [X, u^\perp]) = 0.$$

Thus, finally, we get a simplified condition of flatness of the form

$$\rho(\bar{u}^0 - u^\perp, [X,Y]) = 0 \quad \text{and} \quad \rho(\bar{u}^0 - u^\perp, [X, u^\perp]) = 0, \quad X, Y \in W^\perp.$$

We can replace the condition $X, Y \in W^\perp$ to arrive at one flatness condition

$$\rho(u^\perp - \bar{u}^0, [X,Y]) = 0, \quad X, Y \in V^\perp.$$

We summarize this discussion in the following Theorem.

Theorem 2. *Let $dim(G) = 1$ and let $\mathcal{C}$ be the foliation by complex curves of the (open) submanifold M_{gen} of generic points of manifold M. The following properties are equivalent:*

(1) $\Omega_u|_{M_{gen}} = 0$,

(2) $[X,Y] \in W^\perp \oplus Rv,\ v = \frac{\|u^0\|}{\|u^\perp\|}u^\perp + \frac{\|u^\perp\|}{\|u^0\|}u^0, \quad X, Y \in V^\perp,$

(3) *Let* $D_G \subset T(M)$ *be the distribution generated by the brackets* $[X, Y],\ X, Y \in V^{\perp}$. *Then*

$$\rho(X, \tilde{u}) = 0, \quad X \in D_G.$$

(4) $\Omega_\alpha(X, Y) \in \ Hol_{\mathcal{C}} \mod W^{\perp}, \quad X, Y \in W^{\perp}$.

If M_{gen} *is dense in* M*, one can replace* M_{gen} *by* M *in the statement above.*

Note that the distribution $V^{\perp}$ has codimension 1 and the distribution D_G contains $V^{\perp}$.

6. Polarized harmonic mappings

Let $(M, \rho), (N, \sigma), m \geqslant n$ be Riemannian manifolds of dimensions m and n respectively. Let $u \in \mathcal{X}(M)$ be a vector field. For any smooth mapping $\phi : M \to N$ *without critical points* introduce the following polarized energy density at the point $m \in M$:

$$e_u(\phi)(m) = \frac{\|\phi_{*m}(u(m))\|^2_\rho}{\|\phi_{*m}\|^2_{\rho,\sigma}}, \tag{6.1}$$

where $\phi_{*m} : T_m(M) \to T_{\phi(m)}(N)$ is the linear mapping from $T_m(M)$ with metric ρ into $T_{\phi(m)}(N)$ with metric σ. Here we consider ϕ_* as the section of the induced vector bundle $\phi_* \in \Gamma(T^*(M) \otimes \phi^{-1}(TN))$. Correspondingly, we define the total u-polarized energy of the mapping ϕ as

$$E_u(\phi) = \int_M e_u(\phi)(m)dm, \tag{6.2}$$

where the volume element on M is defined by the metric ρ.

Definition 6. *The mapping* $\phi \in C^\infty(M, N)$ *is called a polarized harmonic mapping if it is a point of maximum of the polarized energy functional* $E_u(\phi)$.

Remark 7. Note that the functional of u-polarized energy may have points of minimum since any mapping ϕ that annihilates the vector field u must be a point of absolute minimum since $E_u(\phi) \geq 0$. It is interesting to note that the condition $\phi_*(u) = 0$ is equivalent to the condition that all components of the mapping ϕ are integrals of motion defined by the vector field u. So, we have

Lemma 7. *The points of absolute minimum of the functional of polarized energy* $E_u(\phi)$ *are exactly the integrals of motion of* u.

The existence of points of maximum is not obvious. Nevertheless we have the following local estimate valid for any compact subset $U \subset M$:

$$\int_U e_u(\phi)(m)dm \leqslant \mathrm{Vol}(U) \sup_{m \in U}(\|u(m)\|_\rho).$$

Remark 8. Let $\lambda \in C^\infty(M)$ be any smooth nowhere zero function and let $v(m) = \lambda(m) \cdot u(m)$ be the vector field obtained by dilation by $\lambda(m)$. Then $e_v(\phi)(m) = |\lambda(m)|^2 e_u(\phi)(m)$. By changing the metric ρ into the conformally equivalent metric $\mu(m)\rho$ for a smooth, stricly positive function $\mu \in C^\infty(M)$, we get a new volume element $\mu^{n/2}dm$ and, as a result,

$$e_{v,\mu\rho}(\phi)(m)dm_{\mu\rho} = \lambda^2 \cdot \mu^{n/2+1} e_{u,\rho}(\phi)(m)dm_\rho.$$

Therefore, dilating u by a non-zero positive function λ is equivalent to the changing the metric on M into another that is conformally equivalent.

Example 3. Take $N = \mathbb{R}$ with the standard metric. Then for a function $\phi : M \to \mathbb{R}$ we have

$$e_u(\phi)(m) = \frac{|u \cdot \phi(m)|^2}{\|\nabla\phi(m)\|^2}.$$

Since $u \cdot \phi(m) = \langle u, \nabla\phi\rangle$, the maximum of density of polarized energy is achieved if $\nabla\phi = \lambda(m)u$ for a function $\lambda \in C^\infty(M)$. Denote by $u^\flat \in \Omega^1(M)$ the 1-form corresponding to the vector field $u \in \mathcal{X}(M)$ via linear isomorphism $T_m(M) \approx T^*_m(M)$ defined by the metric ρ. Then the vector field $\lambda(m)u$ is (locally) the gradient of a function if and only if $d(\lambda \cdot u^\flat) = 0$. A necessary (but not sufficient) condition for the existence of the integrating factor is the Frobenius condition $du^\flat \wedge u^\flat = 0$.

Example 4. Take $M = S^1$, parametrized by the angle θ, and let $u(\theta) = f(\theta)\partial_\theta$. Then $e_u(\phi) = f(\theta)^2$. So all the mappings are points of maximum and minimum at the same time. This illustrates the purpose of the condition that $m \geq n$.

7. Hamiltonian systems on almost Kähler Manifolds

Let (M, J, ω) be a positive almost Kähler manifold. Here J is an almost complex structure on the manifold M, ω is a symplectic form on M such that the symmetric bilinear form $\rho(w, v) = \omega(v, Jw)$ is positive definite (so ρ is a Riemannian metric).

Let $\Phi : G \to \mathrm{Diff}_M$ be a left action of a Lie group G that preserves both complex and symplectic structures (and thus also preserves ρ.) Endow G with a left invariant metric σ and fix a σ-orthonormal basis $\{X_i\}, i = 1, \ldots, k$, of the Lie algebra $\mathfrak{g}$. Denote by $\tilde{X}_i$ the corresponding left invariant vector fields on G. Finally, let $u = \xi_H = J\nabla H$ be the Hamiltonian vector field with Hamiltonian function H, and suppose that H is G-invariant (i.e. $H(gm) = H(m)$ for all $m \in M, g \in G$). Under the hypothesis that G is compact and abelian, we have the following Theorem.

Theorem 3. *Let G be compact and abelian (i.e. a k-dimensional torus), acting on M by complex symplectic diffeomorphisms, let $u = J\nabla H$ be a Hamiltonian vector field with Hamiltonian H an invariant harmonic function on M. Then a mapping*

$$\phi = \exp\left(\sum_{i=1}^{i=k} l_i(m)X_i\right) : M \to G$$

is a polarized harmonic mapping if $\nabla l_i = \lambda_i \cdot J\nabla H, i = 1 \ldots k$ for some $\lambda_i \in C^\infty(M)$. This maximum value of the polarized energy functional is equal to $\int_M \rho(\xi_H, \xi_H)dm$.

Proof. Let $\phi : M \to G$ be a smooth mapping, $m \in M$ an arbitrary point and $u \in \mathcal{X}(M)$ a vector field in M. To calculate $\phi_{*m}(u(m))$, we write ϕ as an exponential

$$\phi(m) = \exp(l^i(m)X_i).$$

Let $t \to g^t m$ be the phase curve u passing through m. Then

$$\phi_{*m}(u(m)) = \left.\frac{d}{dt}\right|_{t=0} \phi(g^t m).$$

For small t,

$$\begin{aligned}\phi(g^t m) &= \exp(l^i(m + tu(m) + o(t))X_i) \\ &= \exp(l^i(m)X_i + t(u \cdot l^i(m))X_i + o(t)).\end{aligned}$$

Recall now the formula (see e.g. [7])

$$\exp(X + tY)\exp(-X) = \exp(t\alpha(adX)Y + o(t)), \quad \alpha(s) = \frac{e^s - 1}{s}.$$

Taking the inverse of this formula with replacements $X \mapsto -X, Y \mapsto -Y$ we get

$$\exp(-X)\exp(X + tY) = \exp(-t\alpha(adX)Y + o(t)).$$

Applying this formula for $X = l^i(m)X_i, Y = (u \cdot l^i(m))X_i$, we get

$$\begin{aligned}\phi(g^t m) &= \exp\big(l^i(m)X_i\big)\exp\big(-t(u \cdot l^i(m))\alpha(ad(l^j(m)X_j))X_i + o(t)\big) \\ &= L_{\phi(m)}\exp\big(-t(u \cdot l^i(m))\alpha(ad(l^j(m)X_j))X_i + o(t)\big).\end{aligned}$$

Taking the derivative by t at $t = 0$ we get

$$\phi_{*m}(u(m)) = -L_{\phi(m)*e}((u \cdot l^i(m))\alpha(ad(l^j(m)X_j))X_i).$$

Since $G = T^k$ is a torus, $\alpha(X) = Id_{\mathfrak{g}}$ and the last formula take simple form

$$\phi_{*m}(u(m)) = -L_{\phi(m)*e}((u \cdot l^i(m))X_i) = -(u \cdot l^i(m))\tilde{X}_{i\phi(m)}.$$

In particular,

$$\begin{aligned}\phi_{*m}(u(m)) &= \sum_{i=1}^{k}\rho(\nabla l_i(m), u(m))\tilde{X}_{i\phi(m)} \\ &= \sum_{i=1}^{k}\rho(\nabla l_i(m), J\nabla H(m))\tilde{X}_{i\phi(m)}.\end{aligned}$$

Using the orthogonality of the left invariant vector fields $\tilde{X}_i$ with respect to the left-invariant metric σ on G we get

$$\|\phi_{*m}(u(m))\|^2_\sigma = \sum_{i=1}^{k}|\rho(\nabla l_i(m), J\nabla H(m))|^2.$$

Therefore at a point $m \in M$,

$$\begin{aligned}e_u(\phi)(m) &= \frac{\|\phi_{*m}(u(m))\|^2_\sigma}{\|\phi_{*m}\|^2_{\rho,\sigma}} \\ &= \frac{\sum_{i=1}^{k}|\rho(\nabla l_i(m), J\nabla H(m))|^2}{\sup_{v\neq 0}\frac{\|\phi_*(v)\|^2}{\|v(m)\|^2}}.\end{aligned}$$

Using the expression for the image of a vector field under the mapping ϕ_* and the fact that the vector fields $\tilde{X}_i$ form an σ-orthonormal basis at each point $g \in T^k$ we get, for all non-critical points of the Hamiltonian H,

$$
\begin{aligned}
\|\phi_*\|^2 &= \sup_{v\neq 0} \frac{\|\phi_*(v)\|^2}{\|v(m)\|^2} \\
&= \sup_{v\neq 0} \frac{\|\sum_i (v\cdot l_i)\tilde{X}_i\|^2}{\rho(v,v)} \\
&= \sup_{v\neq 0} \frac{\sum_i (v\cdot l_i)^2}{\rho(v,v)} \\
&= \sup_{v\neq 0} \frac{\sum_i \rho(v,\nabla l_i)^2}{\rho(v,v)} \geqslant \frac{\sum_i \rho(J\nabla H,\nabla l_i)^2}{\rho(J\nabla H, J\nabla H)}.
\end{aligned}
\tag{7.1}
$$

Combining (7.1) with the expression (6.1) for the energy density $e_u(\phi)(m)$ we arrive at the estimate:

$$e_u(\phi)(m) \leqslant \rho(J\nabla H, J\nabla H).$$

Following these arguments in the opposite order we see that at a non-critical point m of the Hamiltonian H, the functional $e_u(\phi)(m)$ of the mapping ϕ has a maximum at a mapping $\phi(m) = \exp(\sum_i l_i(m)X_i)$ if and only if the linear mapping $A_m : T_m(M) \to T_{\phi(m)}(G)$ given by $v \mapsto \sum_i \rho(v,\nabla l_i)\tilde{X}_i$ realizes its maximum norm $\sup_v \frac{\|A_m v\|}{\|v\|}$ in the direction of $u(m)$, i.e. $v = \lambda(m)J\nabla H(m)$. Geometrically, this means that the ellipsoid $\{A_m v : \|v\| = 1\}$ has the direction of $u(m)$ as the direction of its major axis.

Thus, for $\nabla l_i = \lambda_i J\nabla H$ we get:

$$
\begin{aligned}
e_u(\phi)(m) &= \frac{\sum_{i=1}^k \lambda_i^2 \rho(J\nabla H, J\nabla H)^2}{\sup_{v\neq 0} \frac{\sum_i \lambda_i^2 \rho(J\nabla H, v)}{\rho(v,v)}} \\
&= \frac{\rho(J\nabla H, J\nabla H)^2}{\sup_{v\neq 0} \frac{\rho(J\nabla H, v)^2}{\rho(v,v)}} \\
&= \rho(J\nabla H, J\nabla H),
\end{aligned}
$$

since the supremum of this expression is achieved for the tangent vectors of the form $v = \lambda \cdot J\nabla H$ at each nonsingular point $m \in M$. Therefore, mappings defined as in the formulation of Theorem 3 are points of absolute maximum of the polarized energy in the domain $D \subset M$, given by the formula

$$\sup_{\phi} E_u(\phi) = \int_D \rho(\xi_H, \xi_H)dm.$$

Finally, we address the existence of the functions $l^i(m)$ defining the mapping ϕ. Since H is harmonic, it is the real part of a holomorphic function. Taking l to be the conjugate function to H, the Cauchy-Riemann equations $\nabla l = J\nabla H$ are satisfied. □

Remark 9. We have many choices for the gradients $\nabla l_i = \lambda_i J\nabla H$ that deliver the absolute maximum for energy density.

Remark 10. The arguments above are valid for any Poisson manifold with Poisson tensor P and Hamiltonian vector field $u = P \cdot dH$ with Hamiltonian function H.

Remark 11. Since $\lambda_i(m)J\nabla H = \nabla l_i$ for $i = 1, \dots k$, there are relations $\lambda_j \nabla l_i = \lambda_i \nabla l_j$ for $i \neq j$. This implies that l_i is a function of l_j and vice versa. Thus, for $k > 1$, the mappings ϕ in Theorem 3 are degenerate. Furthermore, if $\phi_0 = \exp(l_0(m)X) : M \to S^1$, $\nabla l_0 = J\nabla H$ is a polarized harmonic mapping, then all other polarized harmonic mappings $\phi : M \to S^1$ have the form $\phi = \exp(l(m)X), \nabla l = \lambda J \nabla H$ with $\lambda(m) = f(l_0(m))$ with some function f.

Lemma 8. *Let $k = 1$. Denote by $\hat{X}$ the generator of an action of S^1 on M. Then the mapping $\phi = \exp(l(m)X) : M \to S^1$, $\nabla l = \lambda J \nabla H$ defines an isometry along S^1 orbits if and only if $\lambda(m) = \frac{1}{\rho(\xi_H, \hat{X})}$.*

Proof. This follows immediately from the condition $\sigma(\phi_{*m}(\hat{X}_m), \phi_{*m}(\hat{X}_m) = \sigma(\tilde{X}, \tilde{X})$ and the G-invariance of both metrics. □

From this Lemma and Remark 11, we see that if $\frac{1}{\rho(\xi_H, \hat{X})} = f(Im(\mathcal{H}))$ for some function f, the polarized harmonic mapping $\phi : M \to S^1$ defines an isometry along the orbits of the S^1-action. In [9] this result was applied to the choice of centroidal rotational frame for a system of point vortices. It was shown there that the choice of $\phi(m)$ as the average angle of the rotated system leads to interesting geometric simplifications in the representation of trajectories of point vortices. In particular, in the rotating frame defined by this average angle, so-called "asymptotic symmetry" in highly chaotic regions of the phase space is explicitly observed.

8. Example: Point vortices in the plane

We now consider a well known problem from fluid mechanics, the motion of N point vortices in the plane. See [1, 2] for details and descriptions of the problem. We begin with N points in the plane $(x_1, y_1), \dots, (x_N, y_N)$ at which there are singular vortices with circulations $\Gamma_1, \dots, \Gamma_N$ respectively. The equations of motion for point vortex k will be

$$\frac{dx_k}{dt} = -\frac{1}{2\pi} {\sum_j}' \frac{\Gamma_j (y_k - y_j)}{r_{jk}^2} \quad \text{and} \quad \frac{dy_k}{dt} = \frac{1}{2\pi} {\sum_j}' \frac{\Gamma_j (x_k - x_j)}{r_{jk}^2}.$$

where $r_{jk}^2 = (x_j - x_k)^2 + (y_j - y_k)^2$ and the primed sigma $\sum'$ indicates that one should omit $j = k$ from the summation. It is simple to verify that this is a Hamiltonian system on $\mathbb{R}^{2N}$ with symplectic form $\omega = \sum_j \Gamma_j dx_j \wedge dy_j$ with Hamiltonian function

$$H = -\frac{1}{4\pi} {\sum_{j,k}}' \Gamma_j \Gamma_k \log r_{jk}.$$

The Hamiltonian equations take the form

$$\Gamma_k \frac{dx_k}{dt} = \frac{\partial H}{\partial y_k} \quad \text{and} \quad \Gamma_k \frac{dy_k}{dt} = -\frac{\partial H}{\partial x_k}.$$

This problem may be rephrased as a Hamiltonian system on the Kähler manifold $\mathbb{C}^N$ with coordinates $z_\alpha = x_\alpha + iy_\alpha, \bar{z}_\alpha = x_\alpha - iy_\alpha$ and symplectic form

$$\omega = \frac{i}{2} \sum_\alpha \Gamma_\alpha dz_\alpha \wedge d\bar{z}_\alpha \tag{8.1}$$

with Hamiltonian

$$H = -\frac{1}{4\pi} {\sum_{\alpha,\beta}}' \Gamma_\alpha \Gamma_\beta \log |z_\alpha - z_\beta|. \tag{8.2}$$

The equations of motion then take the complex form

$$\dot{z}_\alpha = \frac{2}{\Gamma_\alpha} \frac{\partial H}{\partial \bar{z}_\alpha},$$

which is the coordinate statement of the invariant form $X_H = J\nabla H$ where J is the complex structure.

Notice that the Lie group S^1 acts on $\mathbb{C}^N$ via the diagonal action

$$e^{it} \cdot (z_1, \dots, z_N) \mapsto (e^{it} z_1, \dots, e^{it} z_N)$$

and that this action preserves the symplectic form (8.1) and the complex structure. It is also easy to see that the Hamiltonian (8.2) is invariant under this action.

Let $\partial/\partial t$ be the orthonormal basis for the Lie algebra of S^1 given by the standard chart $t \mapsto e^{it}$. As in Theorem 3, the mapping

$$\varphi : \mathbb{C}^N \to S^1 \quad z \mapsto \exp\left(l(z) \frac{\partial}{\partial t} \right) = e^{il(z)}$$

is a polarized harmonic mapping if l is a function that satisfies $\nabla l = J\nabla H$. Since the Hamiltonian $H = -\frac{1}{4\pi} {\sum_{\alpha,\beta}}' \Gamma_\alpha \Gamma_\beta \log |z_\alpha - z_\beta|$ is the real part of the holomorphic function $\mathcal{H} = \frac{1}{4\pi} {\sum_{\alpha,\beta}}' \Gamma_\alpha \Gamma_\beta \log(z_\alpha - z_\beta)$ the function

$$l = \operatorname{Im}(\mathcal{H}) = -\frac{1}{4\pi} {\sum_{\alpha,\beta}}' \Gamma_\alpha \Gamma_\beta \arg(z_\alpha - z_\beta)$$

satisfies the Cauchy-Riemann equations $\nabla l = J\nabla H$. Thus we conclude that the mapping

$$l : \mathbb{C}^N \to S^1 \quad l(z_1, \dots, z_N) = e^{-\frac{i}{4\pi} {\sum_{\alpha,\beta}}' \Gamma_\alpha \Gamma_\beta \arg(z_\alpha - z_\beta)}$$

is a point of absolute maximum for the energy functional E_{X_H} and thus a polarized harmonic mapping for the Hamiltonian vector field X_H. The expression for this mapping may be simplified; using $i \arg(z_\alpha - z_\beta) = \log(z_\alpha - z_\beta) - \log |z_\alpha - z_\beta|$ we obtain

$$\begin{aligned} -\frac{i}{4\pi} {\sum_{\alpha,\beta}}' \Gamma_\alpha \Gamma_\beta \arg(z_\alpha - z_\beta) &= -\frac{1}{4\pi} {\sum_{\alpha,\beta}}' \Gamma_\alpha \Gamma_\beta \big(\log(z_\alpha - z_\beta) - \log |z_\alpha - z_\beta| \big) \\ &= {\sum_{\alpha,\beta}}' \frac{-\Gamma_\alpha \Gamma_\beta}{4\pi} \log(z_\alpha - z_\beta) + {\sum_{\alpha,\beta}}' \frac{\Gamma_\alpha \Gamma_\beta}{4\pi} \log |z_\alpha - z_\beta|. \end{aligned}$$

Thus

$$\begin{aligned} e^{-\frac{i}{4\pi} \sum'_{\alpha,\beta} \Gamma_\alpha \Gamma_\beta \arg(z_\alpha - z_\beta)} &= e^{\sum'_{\alpha,\beta} \frac{-\Gamma_\alpha \Gamma_\beta}{4\pi} \log(z_\alpha - z_\beta) + \sum'_{\alpha,\beta} \frac{\Gamma_\alpha \Gamma_\beta}{4\pi} \log |z_\alpha - z_\beta|} \\ &= \frac{e^{\sum'_{\alpha,\beta} \frac{\Gamma_\alpha \Gamma_\beta}{4\pi} \log |z_\alpha - z_\beta|}}{e^{\sum'_{\alpha,\beta} \frac{\Gamma_\alpha \Gamma_\beta}{4\pi} \log(z_\alpha - z_\beta)}} \\ &= {\prod_{\alpha,\beta}}' \left(\frac{|z_\alpha - z_\beta|}{z_\alpha - z_\beta} \right)^{\frac{\Gamma_\alpha \Gamma_\beta}{4\pi}}. \end{aligned}$$

We summarize this discussion in the following Proposition:

Proposition 6. *Consider the Kähler manifold $\mathbb{C}^N$ with usual complex structure J, symplectic form $\omega = \frac{i}{2}\sum_\alpha \Gamma_\alpha dz_\alpha \wedge d\bar{z}_\alpha$ and Hamiltonian vector field X_H associated to the Hamiltonian $H = -\frac{1}{4\pi}{\sum_{\alpha,\beta}}' \Gamma_\alpha \Gamma_\beta \log|z_\alpha - z_\beta|$. The mapping $l : \mathbb{C}^N \to S^1$ given by*

$$l(z_1, \ldots, z_N) = {\prod_{\alpha,\beta}}' \left(\frac{|z_\alpha - z_\beta|}{z_\alpha - z_\beta} \right)^{\frac{\Gamma_\alpha \Gamma_\beta}{4\pi}}$$

is then a point of maximum of the functional E_{X_H}. That is, l is a polarized harmonic mapping for the vector field X_H.

It remains to discuss mappings which yield minimum values for the functional E_{X_H}. As previously discussed, a mapping will yield a minimum if and only if its components in local coordinates are integrals of motion of the polarizing vector field. The existence of three integrals of motion (in addition to the Hamiltonian) for the point vortex problem is well known (see [1, 10]). These integrals are

$$Q = \sum_\alpha \Gamma_\alpha (z_\alpha + \bar{z}_\alpha) \quad P = \sum_\alpha \Gamma_\alpha (z_\alpha - \bar{z}_\alpha) \quad R = \sum_\alpha \Gamma_\alpha |z_\alpha|^2.$$

Since a fourth integral is known generally not to exist, we may use these integrals to construct all minima of the functional E_{X_H}.

9. Euler-Lagrange equations

In this section we study the Euler-Lagrange equations associated to the polarized energy functional (6.2). Throughout, we employ the notation used in [3]. Fix a mapping $\phi : M \to N$ and consider a 1-parameter variation $\phi_t : M \to N$ such that ϕ_0 coincides with the mapping ϕ. Introduce the vector field $\nu = \dot{\phi}_0 \in C^\infty(\phi^{-1}(TN))$ along the variation, where the vector bundle $\phi^{-1}(TN)$ is endowed with the metric induced by (N, σ) and the corresponding Levi-Civita connection $\nabla^{\phi^{-1}(TN)}$.

Differentiating the variation in the functional we have

$$\partial_t \Big|_{t=0} E_u(\phi_t) = \int_M \partial_t \Big|_{t=0} e_u(\phi_t) dm,$$

and

$$\begin{aligned} \partial_t \Big|_{t=0} e_u(\phi_t) &= \partial_t \Big|_{t=0} \frac{\|\phi_{t*m}(u(m))\|_\rho^2}{\|\phi_{t*m}\|^2} \\ &= \|\phi_{*m}\|^{-2} \left(\partial_t \Big|_{t=0} \|\phi_{t*m}(u(m))\|_\rho^2 - e_u(\phi) \cdot \partial_t \Big|_{t=0} \|\phi_{t*m}\|^{-2} \right). \end{aligned}$$

We may write

$$\begin{aligned} \partial_t \Big|_{t=0} \|\phi_{t*m}\|_\rho^2 &= 2 \langle d\nu, \phi_* \rangle \\ &= -d^* \left(i_\nu \phi^{-1}\sigma \circ \phi_* \right) - \phi^{-1}\sigma\left(\nu, \tau_\phi\right), \end{aligned} \tag{9.1}$$

where $d\nu = \nabla^{\phi^{-1}TN}\nu$ is the covariant differential of a section $\nu \in \Gamma(M, \phi^{-1}(TN)$, [5]. After integration over M the first term after the last equality in (9.1) moves to the boundary and, if ν has a compact support, gives no input into the equation.

Following [3], we have

$$2\left\langle d\nu(u), d\phi(u)\right\rangle_{\phi^{-1}\sigma} = 2u\cdot\left\langle \nu, d\phi(u)\right\rangle_{\phi^{-1}\sigma} - 2\left\langle \nu, \nabla_u^{\phi^{-1}TN} d\phi(u)\right\rangle_{\phi^{-1}\sigma}.$$

As a result

$$\begin{aligned}
&\partial_t\Big|_{t=0} E_u(\phi_t)\\
&= \int_M \|\phi_{*m}\|^{-2}\Big(2u\cdot\left\langle \nu, d\phi(u)\right\rangle_{\phi^{-1}\sigma} - 2\left\langle \nu, \nabla_u^{\phi^{-1}TN} d\phi(u)\right\rangle\\
&\qquad\qquad + e_u(\phi)d^*\left(i_\nu\phi^{-1}\sigma\circ\phi_*\right) + e_u(\phi)\phi^{-1}\sigma\left(\nu,\tau_\phi\right)\Big)dm\\
&= \int_M \mathcal{L}_u\Big(2\|\phi_{*m}\|^{-2}\left\langle \nu, d\phi(u)\right\rangle_{\phi^{-1}\sigma} dm\Big) - 2\|\phi_{*m}\|^{-2} div(u)\left\langle \nu, d\phi(u)\right\rangle_{\phi^{-1}\sigma} dm\\
&\qquad\qquad - 2\left\langle \nu, d\phi(u)\right\rangle \mathcal{L}_u\left(\|\phi_{*m}\|^{-2}\right)dm - 2\|\phi_{*m}\|^{-2}\left\langle \nu, \nabla_u^{\phi^{-1}TN} d\phi(u)\right\rangle dm\\
&= \int_M \Big(\mathcal{L}_u(2\|\phi_{*m}\|^{-2}\langle \nu, d\phi(u)\rangle dm) - 2\Big\langle \nu, \|\phi_{*m}\|^{-2} div(u) d\phi(u)\\
&\quad + \nabla_u^{\phi^{-1}TN}\left(\frac{d\phi(u)}{\|\phi_{*m}\|^2}\right)\Big\rangle\Big)dm + \int_M \frac{e_u(\phi)}{\|\phi_{*m}\|^2}\left(\phi^{-1}(\sigma)(\nu,\tau_\phi) + d^*(i_\nu\phi^{-1}\sigma\circ\phi_*)\right)dm.
\end{aligned}$$

Here we use $\mathcal{L}_u f = u\cdot f = \nabla_u f$ for the Lie derivative with respect to u. Integrating the Lie derivative gives zero if ν has compact support. The same is true for the term containing the coderivative d^*. Therefore we get the following expression for the first variation of the energy functional:

$$2\int_M \langle \nu, \frac{e_u(\phi)}{\|\phi_{*m}\|^2}\tau_\phi - \frac{div(u)}{\|\phi_{*m}\|^2} d\phi(u) - \nabla_u^{\phi^{-1}TN}(\frac{d\phi(u)}{\|\phi_{*m}\|^2})\rangle_{\phi^{-1}(\sigma)} dm.$$

As a result the Euler-Lagrange equation has the form

$$\frac{e_u(\phi)}{\|\phi_{*m}\|^2}\tau_\phi - \frac{div(u)}{\|\phi_{*m}\|^2} d\phi(u) - \nabla_u^{\phi^{-1}TN}(\frac{d\phi(u)}{\|\phi_{*m}\|^2}) = 0, \tag{9.2}$$

or

$$e_u(\phi)\tau_\phi - div(u) d\phi(u) - \nabla_u^{\phi^{-1}TN} d\phi(u) + \frac{u\cdot\|\phi_{*m}\|^2}{\|\phi_{*m}\|^2} d\phi(u) = 0.$$

The Euler-Lagrange equation (9.2) has two obvious classes of solutions:

(1) Points of minimum, $d\phi(u) = 0$. These are integrals of motion of u.
(2) If $div(u) = 0$ and ϕ is a harmonic mapping satisfying the condition that $\dfrac{d\phi(u)}{\|\phi_{*m}\|^2}$ is covariantly constant along u.

Corollary 5. *Let $div(u) = 0$ and let $\phi : M \to N$ be a harmonic mapping. Then ϕ satisfies the Euler-Lagrange equation* (9.2) *if and only if the section $\frac{d\phi(u)}{\|\phi_{*m}\|^2}$ of the bundle $\phi^{-1}TN$ is covariantly constant along the vector field u with respect to the induced connection.*

Using local coordinates $(x^i,\ \frac{\partial}{\partial x^i},\ dx^i)$ and the metric with the Levi-Civita connection $(g_{ij}, {}^M\Gamma^i_{jk})$ on M and $(u^\alpha,\ \frac{\partial}{\partial u^\alpha},\ du^\alpha)$, $(h_{\alpha\beta}, {}^N\Gamma^\alpha_{\beta\gamma})$ on N we write the mapping $\phi : M \to N$ in coordinate form $u^\alpha = \phi^\alpha(x)$, $\phi_* = d\phi(x) = \phi^\alpha_i dx^i \otimes \frac{\partial}{\partial u^\alpha}$. After some transformations we get

$$d\phi(v) = \left[\frac{v^i v^j}{\|d\phi\|^2}\phi^\gamma_{ij} + \phi^\gamma_j v \cdot \left(\frac{v^j}{\|d\phi\|^2}\right) + \frac{\phi^\alpha_j v^j}{\|d\phi\|^2}\phi^\beta_i v^i \ {}^N\Gamma^\gamma_{\beta\alpha}\right]\frac{\partial}{\partial u^\alpha}.$$

Using the expression for the Euler operator of the usual energy functional, [3, Section 2.5], we get as the principal part of the operator

$$\left[\frac{e_v(\phi)}{\|\phi_*\|^2}g^{ij} - v^i v^j\right]\frac{\partial}{\partial x^i}\frac{\partial}{\partial x^j},$$

and the symbol at the non-singular points $x \in M$

$$\sigma(x)(\xi,\xi) = e_v(\phi)g^{ij}\xi_i\xi_j - \langle v,\xi\rangle^2.$$

The quadratic form $e_v(\phi)g^{ij} - v^i v^j$ is negative definite at the points where the energy density is zero.

Consider the flat case, where $\phi : \mathbb{R}^n \to \mathbb{R}^m$. Rewrite the symbol in the form

$$\sigma(x)(\xi,\xi) = \left[e_v(\phi) - \frac{\langle v,\xi\rangle^2}{|\xi|^2}\right]|\xi|^2.$$

We see that in the directions orthogonal to v the symbol is positive definite while along the direction $\xi = v(x)$ of this field we have

$$\sigma(x)(\xi,\xi) = \left[e_v(\phi) - |v(x)|^2\right]|v(x)|^2. \tag{9.3}$$

The expression (9.3) is always nonpositive, and is zero only if the norm of the tangent operator ϕ_* is achieved in the direction of v. We now summarize this discussion in the following

Proposition 7. *The Euler-Lagrange equation for the u-polarized energy functional is hyperbolic and the direction of propagation coincides with the direction of the vector field u.*

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