Group foliation of differential equations using moving frames

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Abstract

We incorporate the new theory of equivariant moving frames for Lie pseudo-groups into Vessiot's method of group foliation of differential equations. The automorphic system is replaced by a set of reconstruction equations on the pseudo-group jets. The result is a completely algorithmic and symbolic procedure for finding both invariant and non-invariant solutions of differential equations admitting a symmetry group.

1 Introduction

The method of group foliation (also called group splitting, or group stratification) is a procedure for obtaining solutions of differential equations invariant under a symmetry group. The idea was proposed by Lie, [27], and subsequently developed by Vessiot, [56]. Later work of Johnson, Ovsiannikov, and others, [9, 19, 47], showed renewed interest. More recently, group foliation has been used to study equations of mathematical physics, [30, 34], and reformulated using the language of exterior differential systems, [3], demonstrating potential for further development and application.

Consider a differential equation $\Delta = 0$ with symmetry group $\mathcal{G}$, possibly infinite dimensional. The method of group foliation uses a foliation of the solution space of $\Delta = 0$ by the orbits of the group action to decompose $\Delta = 0$ into two alternative systems of differential equations, called the *resolving* and *automorphic* systems. An automorphic system, characterized by the property that all solutions are situated on a single orbit of $\mathcal{G}$, describes the leaves of the foliation. The resolving system links the original differential equation to a specific automorphic system in the sense that each resolving system solution specifies a leaf of the foliation. See Figure 1 for the geometry of this construction. Application of group foliation may roughly be understood as a process of removing symmetries; quoting Ovsiannikov, [47]:

> The practical significance of group splitting consists in the fact that solutions of the automorphic system are very simply found at the expense of its automorphic property (by operation with a group transformation on

any of its solutions), and the resolving system turns out to be simple when compared with the initial equation $\Delta = 0$. The latter occurs because the resolving system has fewer solutions than $\Delta = 0$ does because of removal of those excesses which were introduced by the existence of the admitted group $\mathcal{G}$.

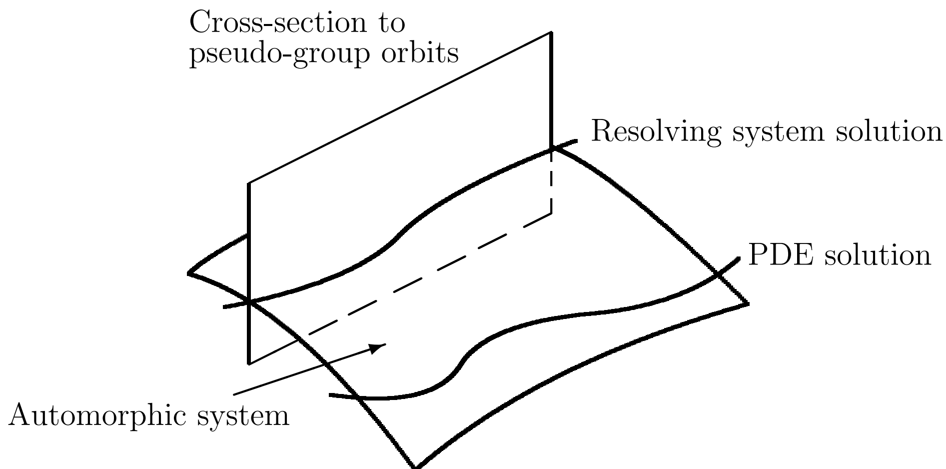


Figure 1: The geometry of group foliation.

Our main tool will be the theory of equivariant moving frames, [13, 43, 44]. The determination of the resolving system relies on the classification of differential invariants and their syzygies, which may be performed algorithmically using the *universal recurrence relation* (3.16). The resolving system may be interpreted as a projection of the original differential equation into a space of invariants, accomplished through the application of a right moving frame. The automorphic system then provides a method for reconstructing solutions to the original differential equation from resolving system solutions. Geometrically, this reconstruction process is the reversal of the right moving frame projection, accomplished by application of a left moving frame.

Our approach was inspired by Mansfield's use of equivariant moving frames to solve ordinary differential equations, [28, Chapter 7]. This approach works for Lie group actions and relies on the choice of a faithful matrix representation for the group. In this paper we adapt these constructions to infinite-dimensional Lie pseudo-group actions. Central to this adaptation is the introduction of the pseudo-group jet differential expressions which, after pull-back by a moving frame, generalize Cartan's structure equations of a moving frame for Lie group actions, [14], and play the role of Mansfield's "curvature matrix" equation in the reconstruction process. The reconstruction step is also related to the reconstruction procedure appearing in symmetry reduction of exterior differential systems, [2, 3, 48].

In its most general formulation, the group foliation method applies to infinite-dimensional Lie pseudo-group actions, so we begin by reviewing in Section 2 the basics of Lie pseudo-groups. The theory of equivariant moving frames is introduced in Section 3. We begin our discussion of group foliation in Section 4.1. In Section 4.2 we incorporate the moving frame apparatus and obtain a new perspective—in particular a natural geometric approach to the reconstruction step—based on moving frames. The

Lie pseudo-group action

$$X = f(x), \qquad Y = y, \qquad U = \frac{u}{f'(x)},$$

considered in [41, 43] is used as a running example for our constructions. This pseudo-group is also used in [48] to illustrate the method of symmetry reduction of exterior differential systems admitting an infinite-dimensional symmetry group; we reproduce these results in Examples 4.6 and 4.25. In Section 5 the group foliation method is applied to several equations of interest, including a nonlinear wave equation studied by Calogero, [5], the equation of a transonic gas flow, and a nonlinear second order ordinary differential equation. Finally, when a symmetry pseudo-group $\mathcal{G}$ admits a chain of normal sub-pseudo-groups, we explain in Section 6 how the reconstruction procedure splits into a sequence of smaller reconstruction problems.

2 Lie pseudo-groups

Since we work with infinite-dimensional Lie pseudo-group actions we restrict our considerations to the analytic category. Given an analytic m-dimensional manifold M, let $\mathcal{D} = \mathcal{D}(M)$ denote the pseudo-group of all local diffeomorphisms $\varphi\colon M \to M$. For each $n \geq 0$ we denote by $\mathcal{D}^{(n)}$ the subbundle formed by their n^{th} order jets $\mathrm{j}_n\varphi$. Introducing the local coordinates $Z = \varphi(z)$ on $\mathcal{D}^{(0)} = M \times M$, we denote by $z = \widetilde{\boldsymbol{\sigma}}(\mathrm{j}_n\varphi)$ and $Z = \widetilde{\boldsymbol{\tau}}(\mathrm{j}_n\varphi)$ the source and target coordinates of φ. The induced coordinates on $\mathcal{D}^{(n)}$ are $\mathrm{j}_n\varphi = (z, Z^{(n)})$, where $Z^{(n)}$ indicates the derivatives

$$Z^b_A = \frac{\partial^k Z^b}{\partial z^{a_1} \cdots \partial z^{a_k}}, \qquad b = 1, \ldots, m, \quad A = (a_1, \ldots, a_k),$$

of order $0 \leq k \leq n$. A local diffeomorphism $\psi \in \mathcal{D}$ acts on $\mathcal{D}^{(n)}$ by either left or right multiplication:

$$\mathrm{L}_\psi(\mathrm{j}_n\varphi|_z) = \mathrm{j}_n(\psi \circ \varphi)|_z \qquad \text{or} \qquad \mathrm{R}_\psi(\mathrm{j}_n\varphi|_z) = \mathrm{j}_n(\varphi \circ \psi^{-1})|_{\psi(z)}. \tag{2.1}$$

The definition of a pseudo-group $\mathcal{G} \subset \mathcal{D}$ is a natural extension of the concept of a local Lie group action. We refer to [17] for a precise definition.

Definition 2.1. A pseudo-group $\mathcal{G} \subset \mathcal{D}$ is called a *Lie pseudo-group* of order $n^\star \geq 1$ if for all finite $n \geq n^\star$

- the pseudo-group jets $\widetilde{\boldsymbol{\sigma}}\colon \mathcal{G}^{(n)} \to M$ form an embedded subbundle of $\widetilde{\boldsymbol{\sigma}}\colon \mathcal{D}^{(n)} \to M$,
- the projection $\pi^{n+1}_n\colon \mathcal{G}^{(n+1)} \to \mathcal{G}^{(n)}$ is a fibration,
- every local diffeomorphism $\varphi \in \mathcal{D}$ satisfying $\mathrm{j}_{n^\star}\varphi \subset \mathcal{G}^{(n^\star)}$ belongs to $\mathcal{G}$.

The above regularity conditions imply that in some coordinate chart, the subbundle $\mathcal{G}^{(n^\star)}$ is described by a system of $n^{\star\text{th}}$ order differential equations

$$F^{(n^\star)}(z, Z^{(n^\star)}) = 0, \tag{2.2}$$

called the *determining system* of $\mathcal{G}$. For $n \geq n^\star$, $\mathcal{G}^{(n)}$ is described by the prolongation of (2.2).

Example 2.2. As our running example we consider the Lie pseudo-group action

$$X = f(x), \qquad Y = y, \qquad U = \frac{u}{f'(x)}, \qquad f \in \mathcal{D}(\mathbb{R}), \tag{2.3}$$

on $M = \mathbb{R}^3 \setminus \{u = 0\}$, defined by the system of differential equations

$$X_y = X_u = 0, \qquad Y = y, \qquad U = \frac{u}{X_x}. \tag{2.4}$$

As is generally the case, it is preferable to work with the Lie algebra of infinitesimal generators of a Lie pseudo-group. Let $\mathcal{X}(M)$ denote the space of locally defined vector fields on M. In local coordinates we use the notation

$$\mathbf{v} = \sum_{a=1}^{m} \zeta^a(z) \frac{\partial}{\partial z^a} \tag{2.5}$$

to denote a vector field. For $0 \leq n \leq \infty$, let $\mathrm{J}^n TM$ denote the n^{th} order jet bundle of the tangent bundle with local coordinates $\mathrm{j}_n\mathbf{v} = (z, \zeta^{(n)})$. Once more, $\zeta^{(n)}$ denotes the collection of derivatives ζ^a_A, $a = 1, \ldots, m$, $0 \leq \#A \leq n$.

Given a Lie pseudo-group $\mathcal{G}$, let $\mathfrak{g} \subset \mathcal{X}(M)$ denote its Lie algebra consisting of local infinitesimal generators whose flows belong to the pseudo-group. A vector field (2.5) is in $\mathfrak{g}$ if its $n^\star$-jet is a solution of the linear system of partial differential equations

$$L^{(n^\star)}(z, \zeta^{(n^\star)}) = 0, \tag{2.6}$$

called the *infinitesimal determining system* of $\mathfrak{g}$, obtained by linearizing the determining system (2.2) at the identity jet. When $\mathcal{G}$ is the symmetry group of a differential equation, the infinitesimal determining system (2.6) is obtained by implementing Lie's algorithm for determining the infinitesimal symmetry generators, [36].

Example 2.3. The infinitesimal generators of the pseudo-group action (2.3) are

$$\mathbf{v} = \xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y} + \phi \frac{\partial}{\partial u} = a(x) \frac{\partial}{\partial x} - u\, a_x(x) \frac{\partial}{\partial u}, \tag{2.7}$$

where $a(x)$ is an arbitrary analytic function. The coefficients of the vector field (2.7) are solutions to the infinitesimal determining system

$$\xi_y = \xi_u = 0, \qquad \eta = 0, \qquad \phi = -u\, \xi_x, \tag{2.8}$$

obtained by linearizing the determining equations (2.4) at the identity jet $\mathbb{1}^{(1)}$. Relations among higher order vector field jets are obtained by considering the prolongation of (2.8).

Dual to the Lie algebra $\mathfrak{g}$ are the $\mathcal{G}$-invariant Maurer–Cartan forms. Since these play an important role in the sequel we now recall the details of their construction, [41]. Beginning with the diffeomorphism pseudo-group $\mathcal{D}$, we split the differential $d = d_M + d_{\mathcal{G}}$ into its horizontal and group (or vertical/contact) components as it is done in the standard variational bicomplex construction, [1], and observe that this splitting is invariant under the pseudo-group multiplication (2.1). Since the target coordinates Z^a are right-invariant, the horizontal one-forms

$$\sigma^{z^a} = d_M Z^a = \sum_{b=1}^{m} Z^a_b\, dz^b, \qquad a = 1, \ldots, m,$$

are also right-invariant. Let $\mathbb{D}_{Z^1}, \ldots, \mathbb{D}_{Z^m}$ be the dual right-invariant total derivative operators defined by

$$d_M F = \sum_{a=1}^{m} (\mathbb{D}_{Z^a} F)\, \sigma^{z^a}, \qquad \text{for } F\colon \mathcal{D}^{(\infty)} \to \mathbb{R}.$$

Explicitly,

$$\mathbb{D}_{Z^a} = \sum_{b=1}^{m} w_a^b\, \mathbb{D}_{z^b}, \qquad \text{where } (w_a^b) = (Z_a^b)^{-1}, \tag{2.9}$$

and

$$\mathbb{D}_{z^b} = \frac{\partial}{\partial z^b} + \sum_{\#A \geq 0} Z_{A,b} \frac{\partial}{\partial Z_A}, \qquad b = 1, \ldots, m, \tag{2.10}$$

are the total derivative operators on $\mathcal{D}^{(\infty)}$. Then, the right-invariant Maurer–Cartan forms are obtained by successively Lie differentiating the zero order invariant contact forms

$$\mu^a = d_{\mathcal{G}} Z^a = dZ^a - \sum_{b=1}^{m} Z_b^a\, dz^b$$

with respect to (2.9):

$$\mu_A^a = \mathbb{D}_Z^A \mu^a.$$

We denote by $\mu^{(n)}$ the set of right invariant Maurer–Cartan forms of order $\leq n$.

For the implementation of the moving frame method, the coordinate expressions of the Maurer–Cartan forms are not required. It is enough to know that these invariant group forms exist since, in practice, most computations involving the Maurer–Cartan forms can be done symbolically.

Under the inclusion map $i\colon \mathcal{G}^{(\infty)} \hookrightarrow \mathcal{D}^{(\infty)}$ the pulled-back Maurer–Cartan forms $\mu_A^a = i^*(\mu_A^a)$ are no longer linearly independent. In the following, to simplify the notation, we systematically avoid writing pull-backs.

Proposition 2.4. Let $\mathcal{G}$ be a Lie pseudo-group of order $n^\star$. Then for all $n \geq n^\star$, the restricted Maurer–Cartan forms $\mu^{(n)}|_{\mathcal{G}}$ satisfy the n^{th} *order lifted linear relations*

$$L^{(n)}(Z, \mu^{(n)}) = 0, \tag{2.11}$$

obtained from the infinitesimal determining system (2.6) and its prolongation by making the substitutions $z^a \to Z^a$ and $\zeta_A^a \to \mu_A^a$.

Example 2.5. Continuing Example 2.3, the right-invariant Maurer–Cartan forms of the Lie pseudo-group (2.3) satisfy the linear relations

$$\mu_Y^x = \mu_U^x = 0, \qquad \mu^y = 0, \qquad \mu^u = -U\mu_X^x, \tag{2.12}$$

obtained from the infinitesimal determining equations (2.8) by making the substitutions

$$\xi_A \to \mu_A^x, \quad \eta_A \to \mu_A^y, \quad \phi_A \to \mu_A^u \qquad \text{and} \qquad x \to X, \quad y \to Y, \quad u \to U.$$

Linear relations among the higher order Maurer–Cartan forms are obtained by Lie differentiating (2.12) with respect to $\mathbb{D}_X$, $\mathbb{D}_Y$, $\mathbb{D}_U$. It follows that a basis of right-invariant Maurer–Cartan forms is given by $\mu_k = \mu_{X^k}^x$, $k \geq 0$.

By pseudo-group inversion, the preceding discussion also holds for the left multiplication (2.1). Denote the inverse of $Z = \varphi(z)$ by $z = \varphi^{-1}(Z)$. Then the above formulas may be adapted to the left action by interchanging the variables z^a and Z^a. In particular, the left invariant Maurer–Cartan forms are obtained by successively Lie differentiating

$$\overline{\mu}^a = dz^a - \sum_{b=1}^{m} z^a_{Z^b}\, dZ^b,$$

with respect to

$$\mathbb{D}_{z^a} = \sum_{b=1}^{m} W^b_a\, \mathbb{D}_{Z^b}, \qquad \text{where} \qquad (W^b_a) = (z^b_{Z^a})^{-1}, \tag{2.13}$$

so that

$$\overline{\mu}^a_A = \mathbb{D}^A_z \overline{\mu}^a.$$

For the implementation of the group foliation method it will be useful to know the relation between left and right invariant Maurer–Cartan forms. For the order zero Maurer–Cartan forms we find that

$$\overline{\mu}^a = dz^a - \sum_{b=1}^{m} z^a_{Z^b}\, dZ^b = -\sum_{b=1}^{m} z^a_{Z^b}(dZ^b - \sum_{c=1}^{m} Z^b_{z^c}\, dz^c) = -\sum_{b=1}^{m} z^a_{Z^b}\, \mu^b. \tag{2.14}$$

The linear relations among the higher order Maurer–Cartan forms are obtained by Lie differentiating (2.14) with respect to (2.13)

$$\overline{\mu}^a_A = -\sum_{b=1}^{m} \sum_{B \le A} \binom{A}{B} \mathbb{D}^B_z(z^a_{Z^b}) \cdot \mathbb{D}^{A-B}_z(\mu^b). \tag{2.15}$$

For example, for the first order Maurer–Cartan forms we have the relations

$$\overline{\mu}^a_b = \mathbb{D}_{z^b}(\overline{\mu}^a) = -\sum_{b,c=1}^{m} W^c_b(z^a_{Z^b Z^c}\, \mu^b + z^a_{Z^b}\, \mu^b_c).$$

For $\mathcal{G} \subset \mathcal{D}$, the relations between the left and right invariant Maurer–Cartan forms are obtain by restricting (2.14), (2.15) to the determining system (2.2) and the lifted determining equations (2.11).

Example 2.6. For our running example, formula (2.14) reduces to

$$\begin{aligned} \overline{\mu} = \overline{\mu}^X &= -[x_X\, \mu^x + x_Y\, \mu^y + x_U\, \mu^u] = -x_X\, \mu, \\ \overline{\mu}^Y &= -[y_X\, \mu^x + y_Y\, \mu^y + y_U\, \mu^u] = -\mu^y = 0, \\ \overline{\mu}^U &= -[u_X\, \mu^x + u_Y\, \mu^y + u_U\, \mu^u] = \frac{u\, x_{XX}}{x_X}\mu^x - \frac{1}{x_X}\mu^u = \frac{u\, x_{XX}}{x_X}\mu - u\, \mu_X, \end{aligned} \tag{2.16}$$

where we used (2.12) and the determining equations

$$x_Y = x_U = 0, \qquad y = Y, \qquad u = \frac{U}{x_X}.$$

Lie differentiating (2.16) with respect to

$$\mathbb{D}_x = \frac{1}{x_X}\mathbb{D}_X + \frac{u\,x_{XX}}{x_X}\mathbb{D}_U, \qquad \mathbb{D}_y = \mathbb{D}_Y, \qquad \mathbb{D}_u = x_X\mathbb{D}_U$$

yields the relations among the higher order Maurer–Cartan forms. For example,

$$\begin{aligned}\overline{\mu}_x =& \mathbb{D}_x(\overline{\mu}) = -\frac{x_{XX}}{x_X}\mu - \mu_X,\\ \overline{\mu}_{xx} =& \mathbb{D}_x(\overline{\mu}_x) = \left(\frac{x_{XX}^2}{x_X^3} - \frac{x_{XXX}}{x_X^2}\right)\mu - \frac{x_{XX}}{x_X^2}\mu_X - \frac{1}{x_X}\mu_{XX}.\end{aligned}$$

3 Moving frames

We are interested in the action of a Lie pseudo-group $\mathcal{G}$ on p-dimensional submanifolds $S \subset M$, with $1 \leq p < m = \dim M$. To this end, let $\mathrm{J}^n = \mathrm{J}^n(M,p)$ denote the n^{th} order extended jet bundle of equivalence classes of p-dimensional submanifolds under the equivalence relation of n^{th} order contact, [35]. Locally, we identify $M \simeq X \times U$ with the Cartesian product of the submanifolds X and U with local coordinates $z = (x,u)$. The coordinates $x = (x^1,\ldots,x^p)$ and $u = (u^1,\ldots,u^q)$ are considered as independent and dependent variables respectively. This induces the local coordinates $z^{(n)} = (x,u^{(n)})$ on J^n, where $u^{(n)}$ denotes the collection of derivatives u^α_J, with $\alpha = 1,\ldots,q$ and $0 \leq \#J \leq n$.

We introduce the n^{th} *order lifted bundle*

$$\mathcal{E}^{(n)} = \mathrm{J}^n \times_M \mathcal{G}^{(n)} \to \mathrm{J}^n$$

whose local coordinates are given by $(z^{(n)}, g^{(n)})$, where $z^{(n)}$ are the n^{th} order submanifold jet coordinates and $g^{(n)}$ are the fiber coordinates along $\mathcal{G}^{(n)}|_z$. On the infinite order lifted bundle $\mathcal{E}^{(\infty)}$, define the *total derivative operators*

$$D_{x^i} = \mathbb{D}_{x^i} + \sum_{\alpha=1}^{q}\left[u^\alpha_i \mathbb{D}_{u^\alpha} + \sum_J u^\alpha_{J,i}\frac{\partial}{\partial u^\alpha_J}\right], \qquad i = 1,\ldots,p,$$

where the expressions for the differential operators $\mathbb{D}_{x^i}$, $\mathbb{D}_{u^\alpha}$ are given in (2.10).

A Lie pseudo-group $\mathcal{G}$ acts on J^n by the usual prolonged action

$$(X, U^{(n)}) = g^{(n)} \cdot (x, u^{(n)}) = g^{(n)} \cdot \mathrm{j}_n S = \mathrm{j}_n(g \cdot S). \tag{3.1}$$

The coordinate expressions of the prolonged action are obtained by applying the *lifted total derivative operators*

$$D_{X^i} = \sum_{j=1}^{p} B^j_i D_{x^j}, \qquad \text{where} \qquad (B^j_i) = (D_{x^i}X^j)^{-1},$$

to the dependent target coordinates U^α:

$$U^\alpha_{X^J} = D^J_X U^\alpha, \qquad \alpha = 1,\ldots,q, \quad \#J \geq 0.$$

The lifted bundle $\mathcal{E}^{(n)}$ has groupoid structure with source map $\boldsymbol{\sigma}^{(n)}(z^{(n)}, g^{(n)}) = z^{(n)}$ given by the projection onto the first factor and target map $\boldsymbol{\tau}^{(n)}(z^{(n)}, g^{(n)}) = g^{(n)} \cdot z^{(n)}$ given by the prolonged action (3.1). On $\mathcal{E}^{(\infty)}$ we use $\boldsymbol{\sigma}$ and $\boldsymbol{\tau}$ to denote the source and target maps.

Example 3.1. The Lie pseudo-group (2.3) is now assumed to act on surfaces in $\mathbb{R}^3$ locally given as graphs $\{x, y, u(x, y)\}$. In this setting, the prolonged action is obtained by applying the lifted total derivative operators

$$D_X = \frac{1}{f_x} D_x, \qquad D_Y = D_y,$$

to U. For example, the second order prolonged action is

$$U_Y = \frac{u_y}{f_x}, \qquad U_X = \frac{u_x f_x - u f_{xx}}{f_x^3}, \qquad U_{YY} = \frac{u_{yy}}{f_x},$$
$$U_{XY} = \frac{u_{xy} - U_Y f_{xx}}{f_x^2}, \qquad U_{XX} = \frac{u_{xx} f_x - u f_{xxx}}{f_x^4} - 3\frac{U_X f_{xx}}{f_x^2}.$$

From this point forward, we assume that the Lie pseudo-group acts regularly on J^n for all n. This means that the orbits of the pseudo-group action form a regular foliation and its leaves intersect small open sets in pathwise connected subsets.

Definition 3.2. Let

$$\mathcal{G}^{(n)}_{z^{(n)}} = \left\{ g^{(n)} \in \mathcal{G}^{(n)}|_z : g^{(n)} \cdot z^{(n)} = z^{(n)} \right\}$$

be the isotropy subgroup of $z^{(n)}$. The pseudo-group $\mathcal{G}$ acts *freely* at $z^{(n)}$ if $\mathcal{G}^{(n)}_{z^{(n)}} = \{\mathbb{1}^{(n)}_z\}$. The pseudo-group $\mathcal{G}$ is said to act *freely at order* n if it acts freely on an open subset $\mathcal{V}^n \subset \mathrm{J}^n$, called the set of regular n-jets.

Definition 3.3. Let $\mathcal{G}$ be a Lie pseudo-group acting regularly and freely on $\mathcal{V}^n \subset \mathrm{J}^n$ and let $\mathcal{K}^n \subset \mathcal{V}^n$ be a local cross-section to the pseudo-group orbits. Given $z^{(n)} \in \mathcal{V}^n$, the n^{th}-order *right moving frame*

$$\varrho^{(n)}(z^{(n)}) = (z^{(n)}, \rho^{(n)}(z^{(n)}))$$

is the section of the lifted bundle $\mathcal{E}^{(n)}$ where the fiber component $\rho^{(n)}(z^{(n)})$ is the unique n^{th} order pseudo-group jet in $\mathcal{G}^{(n)}|_z$ such that $\boldsymbol{\tau}^{(n)}[\varrho^{(n)}(z^{(n)})] = \rho^{(n)}(z^{(n)}) \cdot z^{(n)} \in \mathcal{K}^n$.

Assuming, to simplify the discussion, that $\mathcal{K}^n$ is the coordinate cross-section

$$x^{i_1} = c_1, \quad \ldots, \quad x^{i_l} = c_l, \qquad u^{\alpha_{l+1}}_{J_{l+1}} = c_{l+1}, \quad \ldots, \quad u^{\alpha_{d_n}}_{J_{d_n}} = c_{d_n}, \tag{3.2}$$

where $d_n = \dim\, \mathcal{G}^{(n)}|_z$ is the fiber dimension of the subbundle $\mathcal{G}^{(n)}$, the corresponding right moving frame is obtained by solving the *normalization equations*

$$X^{i_1}(z^{(n)}, g^{(n)}) = c_1, \quad \ldots, \quad X^{i_l}(z^{(n)}, g^{(n)}) = c_l,$$
$$U^{\alpha_{l+1}}_{X^{J_{l+1}}}(z^{(n)}, g^{(n)}) = c_{l+1}, \quad \ldots, \quad U^{\alpha_{d_n}}_{X^{J_{d_n}}}(z^{(n)}, g^{(n)}) = c_{d_n},$$

for the pseudo-group jets $g^{(n)} = \rho^{(n)}(z^{(n)})$ so that

$$\varrho^{(n)}(z^{(n)}) = (z^{(n)}, \rho^{(n)}(z^{(n)})). \tag{3.3}$$

To each right moving frame (3.3) corresponds a unique *left moving frame* $\overline{\varrho}^{(n)}$ obtained by pseudo-group inversion:

$$\overline{\varrho}^{(n)}(z^{(n)}) = (\rho^{(n)}(z^{(n)}) \cdot z^{(n)}, (\rho^{(n)}(z^{(n)}))^{-1}).$$

In the following, we let $\overline{\rho}^{(n)}(z^{(n)}) = (\rho^{(n)}(z^{(n)}))^{-1}$ denote the inverse of the pseudo-group jet $\rho^{(n)}(z^{(n)})$ so that

$$\overline{\varrho}^{(n)}(z^{(n)}) = (\rho^{(n)}(z^{(n)}) \cdot z^{(n)}, \overline{\rho}^{(n)}(z^{(n)})) \qquad \text{where} \qquad z^{(n)} = \boldsymbol{\tau}^{(n)}(\overline{\varrho}^{(n)}). \tag{3.4}$$

Given a (right) moving frame, there is a systematic procedure for invariantizing differential functions and differential forms. First recall the standard coframe on J^∞ given by the horizontal one-forms

$$dx^1, \ldots, dx^p, \tag{3.5a}$$

and the basic contact one-forms

$$\theta^\alpha_J = du^\alpha_J - \sum_{j=1}^{p} u^\alpha_{J,j}\, dx^j, \qquad \alpha = 1, \ldots, q, \quad \#J \geq 0. \tag{3.5b}$$

Supplementing (3.5) with the Maurer–Cartan forms $\mu^a_A|_{\mathcal{G}}$ yields a coframe for the lifted bundle $\mathcal{E}^{(\infty)}$.

Definition 3.4. Let ω be a differential form on J^∞. Its *lift* is the $\mathcal{G}$-invariant differential form

$$\boldsymbol{\lambda}(\omega) = \pi_{\mathrm{J}}[\boldsymbol{\tau}^*(\omega)], \tag{3.6}$$

where π_{J} is the projection onto jet forms obtained by setting the Maurer–Cartan forms equal to zero.

We denote by

$$\Omega^i = \boldsymbol{\lambda}(dx^i), \qquad \Theta^\alpha_J = \boldsymbol{\lambda}(\theta^\alpha_J), \tag{3.7}$$

the lift of the standard jet coframe. When ω is a submanifold jet coordinate, its lift is just the usual prolonged action:

$$X^i = \boldsymbol{\lambda}(x^i), \qquad U^\alpha_{X^J} = \boldsymbol{\lambda}(u^\alpha_J).$$

The lift map (3.6) may also be extended to the vector field jet $\zeta^{(n)}$ by defining

$$\boldsymbol{\lambda}(\zeta^a_A) = \mu^a_A$$

to be the corresponding right-invariant Maurer–Cartan form.

Definition 3.5. Let $\varrho = \varrho^{(\infty)} \colon \mathrm{J}^\infty \to \mathcal{E}^{(\infty)}$ be a right moving frame, then the *invariantizaton map* $\iota \colon \boldsymbol{\Omega}^*(\mathrm{J}^\infty) \to \boldsymbol{\Omega}^*(\mathrm{J}^\infty)$ is defined by

$$\iota = \varrho^* \circ \boldsymbol{\lambda}. \tag{3.8}$$

We denote by

$$\begin{aligned} \varpi^i &= \varrho^*(\Omega^i) = \iota(dx^i), \qquad && i = 1, \ldots, p, \\ \vartheta^\alpha_J &= \varrho^*(\Theta^\alpha_J) = \iota(\theta^\alpha_J), \qquad && \alpha = 1, \ldots, q, \quad \#J \geq 0, \end{aligned} \tag{3.9}$$

the invariantization of the horizontal coframe and basic contact one-forms. Since the lifted contact forms (3.7) and their invariant counterparts in (3.9) are not used in the group foliation method we introduce the equivalence notation $\equiv$ to denote equality of two differential one-forms up to a lifted or invariant contact form. Finally, we denote the invariantization of the submanifold jet coordinates by

$$H^i = \iota(x^i), \qquad I^\alpha_J = \iota(u^\alpha_J), \tag{3.10}$$

and refer to them as *normalized invariants*. By construction, the invariantization of the jet coordinates (3.2) defining the cross-section $\mathcal{K}^\infty$ are constant, and for this reason are called phantom invariants.

Proposition 3.6. The normalized invariants (3.10) form a complete set of functionally independent differential invariants. In particular, using the invariantization map (3.8) any invariant $J(x, u^{(n)})$ can be expressed as

$$J(x, u^{(n)}) = \iota[J(x, u^{(n)})] = J(H, I^{(n)}).$$

Example 3.7. We now construct a moving frame for the Lie pseudo-group action (2.3). A standard cross-section to the pseudo-group orbits is

$$x = 0, \qquad u = 1, \qquad u_{x^k} = 0, \qquad k \geq 1. \tag{3.11}$$

Solving the normalization equations $U = 1$, $X = U_{X^k} = 0$, $k \geq 1$, for the pseudo-group parameters yields the right moving frame

$$f = 0, \qquad f_{x^k} = u_{x^{k-1}}, \qquad k \geq 1. \tag{3.12}$$

Up to second order, the invariantization of the submanifold jet coordinates produces the normalized invariants

$$\begin{gathered} H^y = \iota(y) = y, \qquad I_{01} = \iota(u_y) = \frac{u_y}{u}, \\ I_{11} = \iota(u_{xy}) = \frac{u u_{xy} - u_x u_y}{u^3}, \qquad I_{02} = \iota(u_{yy}) = \frac{u_{yy}}{u}. \end{gathered} \tag{3.13}$$

The invariantization of the horizontal coframe gives the invariant one-forms

$$\varpi^x = \varrho^*(d_J X) = u\, dx, \qquad \varpi^y = \varrho^*(d_J Y) = dy.$$

The corresponding left moving frame is obtained by inverting the right moving frame (3.12):

$$\bar{f} = x, \qquad \bar{f}_X = \frac{1}{f_x} = \frac{1}{u}, \qquad \bar{f}_{XX} = -\frac{f_{xx}}{f_x^3} = \frac{u_x}{u^3}, \qquad \cdots. \tag{3.14}$$

Theorem 3.8. Let ω be a differential form on J^∞, then

$$d[\boldsymbol{\lambda}(\omega)] = \boldsymbol{\lambda}[d\omega + \mathbf{v}^{(\infty)}(\omega)]. \tag{3.15}$$

An immediate consequence of (3.15) is that

$$d[\iota(\omega)] = \iota[d\omega + \mathbf{v}^{(\infty)}(\omega)]. \tag{3.16}$$

The identity (3.16) is called the *universal recurrence relation*. We are particularly interested in the case when ω is a differential function, and more particularly, when ω

is one of the submanifold jet coordinates. Substituting x^i and u^α_J in (3.16) we obtain the *invariant recurrence relations*

$$dH^i = \varpi^i + M^i, \qquad dI^\alpha_J \equiv I^\alpha_{J,i}\,\varpi^i + N^\alpha_J, \tag{3.17}$$

for the normalized invariants. The correction terms M^i, N^α_J come from the Lie algebraic term $\iota(\mathbf{v}^{(\infty)}(\omega))$ in (3.16). One of most important features of (3.17) or (3.16) is that these equations do not require the coordinate expression of the invariant object to be computed, [43].

Example 3.9. In this example we compute the invariant recurrence relations (3.17) for the normalized invariants (3.13). To compute the lifted recurrence relation (3.15) we need the prolongation of the infinitesimal generator (2.7):

$$\begin{aligned}\mathbf{v}^{(\infty)} =& a(x)\frac{\partial}{\partial x} - u\,a_x\frac{\partial}{\partial u} - (u\,a_{xx} + 2u_x\,a_x)\frac{\partial}{\partial u_x} - u_y\,a_x\frac{\partial}{\partial u_y} - u_{yy}\,a_x\frac{\partial}{\partial u_{yy}}\\ &- (u_y\,a_{xx} + 2u_{xy}\,a_x)\frac{\partial}{\partial u_{xy}} - (u\,a_{xxx} + 3u_x\,a_{xx} + 3u_{xx}\,a_x)\frac{\partial}{\partial u_{xx}} - \cdots .\end{aligned}$$

Substituting x, y, u, u_x, u_y, $\ldots$ for ω in (3.15) we obtain, modulo contact forms,

$$\begin{aligned}
&dX = \Omega^x + \mu, \qquad dY = \Omega^y,\\
&dU \equiv U_X\,\Omega^x + U_Y\,\Omega^y - U\,\mu_X,\\
&dU_X \equiv U_{XX}\,\Omega^x + U_{XY}\,\Omega^y - U\,\mu_{XX} - 2U_X\,\mu_X,\\
&dU_Y \equiv U_{XY}\,\Omega^x + U_{YY}\,\Omega^y - U_Y\,\mu_X,\\
&dU_{XX} \equiv U_{XXX}\,\Omega^x + U_{XXY}\,\Omega^y - U\,\mu_{XXX} - 3U_X\,\mu_{XX} - 3U_{XX}\,\mu_X,\\
&dU_{XY} \equiv U_{XXY}\,\Omega^x + U_{XYY}\,\Omega^y - U_Y\,\mu_{XX} - 2U_{XY}\,\mu_X,\\
&dU_{YY} \equiv U_{XYY}\,\Omega^x + U_{YYY}\,\Omega^y - U_{YY}\,\mu_X, \qquad \ldots .
\end{aligned} \tag{3.18}$$

Pulling-back (3.18) by the right moving frame ρ, the left-hand side of (3.18) is identically zero for the phantom invariants $H^x = 0$, $I = 1$, $I_{k0} = 0$, $k \geq 1$. Solving these equations for the pulled-back Maurer–Cartan forms $\mu_k = \varrho^*\mu_k$, the result is

$$\mu = -\varpi^x, \qquad \mu_k \equiv I_{k-1,1}\,\varpi^y, \qquad k \geq 1. \tag{3.19}$$

Substituting the expressions (3.19) into the remaining recurrence relations (3.18) yields the invariant recurrence relations

$$\begin{gathered}dH^y = \varpi^y, \qquad dI_{01} \equiv I_{11}\,\varpi^x + I_{02}\,\varpi^y - I_{01}^2\,\varpi^y,\\ dI_{11} \equiv I_{21}\,\varpi^x + I_{12}\,\varpi^y - 3I_{01}I_{11}\,\varpi^y, \qquad dI_{02} \equiv I_{12}\,\varpi^x + I_{03}\,\varpi^y - I_{01}I_{02}\,\varpi^y,\end{gathered} \tag{3.20}$$

and so on. Let $\mathcal{D}_i$ be the invariant total differential operators dual to the invariant horizontal one-forms ϖ^i defined by

$$dF \equiv \sum_{i=1}^{p} \mathcal{D}_i(F)\,\varpi^i \qquad \text{for any differential function} \quad F(x, u^{(n)}).$$

Since the invariant horizontal one-forms ϖ^i are linearly independent we deduce from (3.20) the recurrence relations

$$\begin{aligned}
&\mathcal{D}_x I_{01} = I_{11}, \qquad &&\mathcal{D}_y I_{01} = I_{02} - I_{01}^2,\\
&\mathcal{D}_x I_{11} = I_{21}, \qquad &&\mathcal{D}_y I_{11} = I_{12} - 3I_{01}I_{11},\\
&\mathcal{D}_x I_{02} = I_{12}, \qquad &&\mathcal{D}_y I_{02} = I_{03} - I_{01}I_{02},
\end{aligned} \tag{3.21}$$

among the low order normalized invariants. Note that the invariant I_{12} appears twice on the right-hand side of (3.21). Eliminating this invariant we obtain the relation

$$\mathcal{D}_x I_{02} = \mathcal{D}_y I_{11} + 3 I_{01} I_{11}. \tag{3.22}$$

Definition 3.10. A set of invariants $\mathcal{I}$ is said to generate the algebra of differential invariants with respect to the invariant derivative operators $\mathcal{D}_1, \ldots, \mathcal{D}_p$ if all differential invariants can be expressed as some function of the invariants $I \in \mathcal{I}$ and their invariant derivatives $\mathcal{D}_J I$.

The *Fundamental Basis Theorem*—first proved for finite-dimensional group actions by Lie, [26, p. 760], and later extended to infinite-dimensional Lie pseudo-groups by Tresse, [54]—guarantees that the set $\mathcal{I}$ may be taken to be finite. That is, the algebra of differential invariants is generated by a finite number of invariants. Modern proofs of the Fundamental Basis Theorem appear in the textbooks [36, 47]; other proofs based on Spencer cohomology, [23], Weyl algebras, [32], homological methods, [21] or moving frames, [15, 39, 44], also exist.

Using Gröbner basis techniques, the proof of the Basis Theorem presented in [44] is constructive and also identifies a generating set. The proof relies on the assumption that moving frames constructed are of *minimal order*, and hence we assume from now on every moving frame to be of minimal order. Intuitively, a moving frame is of minimal order if during the normalization procedure the pseudo-group parameters are normalized as soon as possible; we refer the reader to [15, 39] for a precise definition.

Understanding functional dependence relations among the invariants will be central to the implementation of the group foliation method.

Definition 3.11. A *syzygy* among the generating differential invariants $\mathcal{I} = \{I^1, \ldots, I^k\}$ is a nontrivial functional relationship

$$S(\ldots, \mathcal{D}_L I^1, \ldots, \mathcal{D}_K I^k, \ldots) = 0$$

among the invariants I^ν and their various invariant derivatives $\mathcal{D}_J I^\nu$.

Example 3.12. Continuing Example 3.9, setting $\omega = dx$ and $\omega = dy$ in the recurrence relation (3.16) we find that

$$d\varpi^x = I_{01}\, \varpi^y \wedge \varpi^x, \qquad d\varpi^y = 0.$$

By duality we deduce the commutation relation

$$[\mathcal{D}_x, \mathcal{D}_y] = I_{01}\, \mathcal{D}_x. \tag{3.23}$$

Syzygies arising from commutation relations such as the above are called *commutator syzygies*. For example, for any differential invariant I one finds by application of (3.23) the syzygy

$$\mathcal{D}_x \mathcal{D}_y I = \mathcal{D}_y \mathcal{D}_x I + I_{01} \mathcal{D}_x I. \tag{3.24}$$

Definition 3.13. A collection $\mathcal{S} = \{S_1, \ldots, S_k\}$ of syzygies is said to form a *generating system* if every syzygy can be written as a linear combination of members of $\mathcal{S}$ and finitely many of their derivatives, modulo the commutator syzygies.

Theorem 3.14. Let $\mathcal{G}$ be a Lie pseudo-group acting locally freely on an open subset of the submanifold jet bundle J^n for some $n \geq 1$. Then the algebra of syzygies is generated by a finite number of fundamental sygygies.

A comprehensive discussion of syzygies and a proof of Theorem 3.14 appears in [44]. The proof is again constructive and based on Gröbner basis methods. We note that in applications it is generally possible to avoid the introduction of the Gröbner basis machinery. The generating sets $\mathcal{I}$ and $\mathcal{S}$ for the algebra of differential invariants and the algebra of syzygies can be found by direct observation.

Example 3.15. Continuing Example 3.9, we conclude from the recurrence relations (3.21) that the second order normalized invariants I_{11} and I_{02} are expressible in terms of the normalized invariants I_{01} and H^y and their invariant derivatives with respect to $\mathcal{D}_x$ and $\mathcal{D}_y$. The same holds for higher order normalized invariants, and we conclude that the algebra of differential invariants of the pseudo-group (2.3) is generated by the normalized invariants I_{01} and H^y and the invariant derivative operators $\mathcal{D}_x = (1/u)D_x$ and $\mathcal{D}_y = D_y$.

Also, there is no fundamental syzygy among the generating invariants $\{H^y, I_{01}\}$. Every syzygy must be trivial modulo the commutator syzygies. For example, substituting $I = I_{01}$ in (3.24) and using the recurrence relations (3.21), we recover the syzygy (3.22).

In the above discussion, the invariant derivative operators $\mathcal{D}_1, \ldots, \mathcal{D}_p$ can be replaced by any other set of p linearly independent invariant total derivative operators. In doing so, the structure of the algebra of differential invariants may change. As the next example shows, the generating set of invariants and fundamental syzygies are dependent on the basis of invariant total derivative operators.

Example 3.16. We now revisit Example 3.15 using a different set of invariant derivative operators. To simplify the notation, let

$$H = H^y, \qquad J = I_{01}, \qquad K = I_{11}, \qquad L = I_{02}. \tag{3.25}$$

From (3.20), we have that

$$dH \wedge dJ \equiv K\, \varpi^y \wedge \varpi^x. \tag{3.26}$$

Working on the open subset of jet space where $K \neq 0$, one can replace the invariant total derivative operators $\mathcal{D}_x$ and $\mathcal{D}_y$ by the invariant Tresse derivatives D_H and D_J, [22]. By the chain rule,

$$\begin{aligned} \mathcal{D}_x &= \mathcal{D}_x H \cdot D_H + \mathcal{D}_x J \cdot D_J = K D_J, \\ \mathcal{D}_y &= \mathcal{D}_y H \cdot D_H + \mathcal{D}_y J \cdot D_J = D_H + (L - J^2) D_J. \end{aligned} \tag{3.27}$$

In terms of these Tresse derivatives, the algebra of differential invariants cannot be generated by the invariants $\{H^y, I_{01}\} = \{H, J\}$ since

$$D_H(H) = D_J(J) = 1 \qquad \text{and} \qquad D_H(J) = D_J(H) = 0.$$

In this case, a generating set of invariants is given by the four normalized invariants (3.25). There is now one fundamental syzygy obtained by expressing (3.22) in terms of the operators (3.27):

$$\mathcal{D}_x L = \mathcal{D}_y K + 3JK \qquad \Longleftrightarrow \qquad K D_J L = D_H K + (L - J^2) D_J K + 3JK. \tag{3.28}$$

4 Group foliation

In the first part of this section we review the classical method of group foliation, mostly following Ovsiannikov's treatment, [47]. Moving frames are used when possible to simplify the constructions. In particular, the derivation of the automorphic and resolving systems is done symbolically without relying on coordinate expressions for the differential invariants. In the second part of this section, the moving frame method is used to obtain a symbolic procedure for reconstructing solutions of the original differential equation from solutions of the resolving system. This reconstruction procedure differs from the classical approach using automorphic systems, which requires explicit formulae for differential invariants.

4.1 Vessiot's group foliation method

Given a differential equation $\Delta = 0$, group foliation splits the problem of solving $\Delta = 0$ into one of solving two associated systems of differential equations called the *resolving* and *automorphic* systems. More precisely, it is an associated family of equations; each solution to the resolving system determines a particular $\mathcal{G}$-automorphic system, which in turns yields solutions to the original equation $\Delta = 0$.

Definition 4.1. A system of differential equations is called *$\mathcal{G}$-automorphic* if all of its solutions can be obtained from a single solution via transformations belonging to $\mathcal{G}$.

We now describe the method rigorously. Suppose that

$$\Delta(x, u^{(n)}) = 0$$

is an n^{th} order differential equation admitting a Lie pseudo-group $\mathcal{G}$ of symmetries. By definition of invariance, $\mathcal{G}$ maps solutions of $\Delta = 0$ to other solutions. Thus, there is an induced action of $\mathcal{G}$ on the solution set, partitioning this space into orbits. If the jets of solutions lie within the set of regular jets $\mathcal{V}^k \subset \mathrm{J}^k$, these orbits determine invariant submanifolds in J^k, $k \geq 0$, traced out by the action of $\mathcal{G}$ on the prolonged graph of a given solution. The description of these invariant submanifolds using the differential invariants of $\mathcal{G}$ leads to the main idea of the group foliation method.

Let $\mathcal{K}^k$ be a cross-section to the prolonged action of $\mathcal{G}$ on J^k and let $u_0 \colon X \to U$ be an arbitrary function whose prolonged graph $(x, u_0^{(k)}(x))$ lies in a neighborhood of $\mathcal{K}^k$. Later on, $u_0(x)$ will be a solution of the differential equation $\Delta = 0$, but the immediate discussion does not rely on this assumption. Consider the orbit under $\mathcal{G}$ of the k^{th} prolongation of the graph $(x, u_0^{(k)}(x))$:

$$\mathcal{A}(u_0^{(k)}) = \{g^{(k)} \cdot (x, u_0^{(k)}(x)) : g^{(k)} \in \mathcal{G}^{(k)}|_{(x,u_0^{(k)})}\} \subset \mathrm{J}^k.$$

Let r_k be the dimension of the intersection of $\mathcal{A}(u_0^{(k)})$ with the cross-section $\mathcal{K}^k$. We assume that the dimension of this intersection is constant. Increasing the order of prolongation, we have the non-decreasing sequence

$$0 \leq r_0 \leq r_1 \leq \cdots \leq p.$$

Definition 4.2. The smallest order s such that $r_s = r_{s+i}$ for all $i \geq 1$ is called the *order* and $r = r_s$ is called the *invariant rank of the function* u_0.

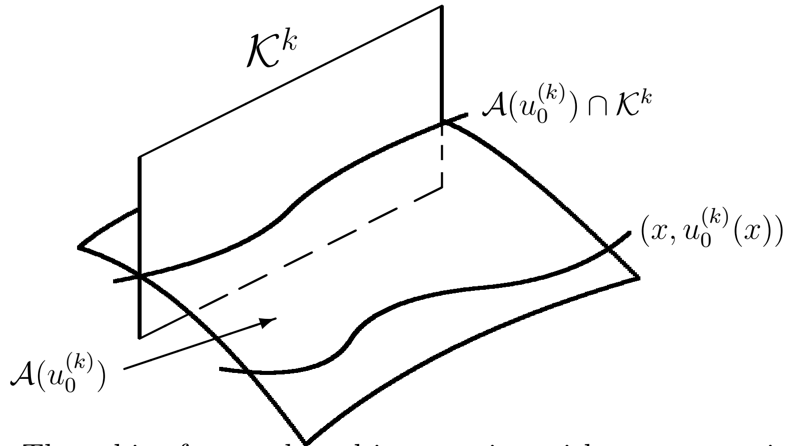


Figure 2: The orbit of a graph and intersection with a cross-section.

As guaranteed by the Fundamental Basis Theorem, we may choose $k \geq s$ so that there is a functionally independent generating set $\mathcal{I}$ of differential invariants of order $\leq k$. These invariants provide coordinates for the cross-section $\mathcal{K}^k$ and allows us to write the intersection $\mathcal{A}(u_0^{(k)}) \cap \mathcal{K}^k$ as a parametrized submanifold of $\mathcal{K}^k$. For this purpose, distinguish a set of functionally independent differential invariants $\{J^1, \ldots, J^r\} \subset \mathcal{I}$, to be used as parametric variables. We may then use the remaining invariants as dependent variables for the parametrization, writing $\mathcal{A}(u_0^{(k)}) \cap \mathcal{K}^k$ locally as a graph in $\mathcal{K}^k$:

$$\mathcal{A}_r : \qquad K^1 = F^1(J^1, \ldots, J^r), \qquad \ldots, \qquad K^\nu = F^\nu(J^1, \ldots, J^r), \tag{4.1}$$

where $\{J^1, \ldots, J^r, K^1, \ldots, K^\nu\}$ is the full generating set of invariants. The system (4.1) is automorphic and will be called an automorphic system $\mathcal{A}_r$ of rank r, dropping reference to u_0. In [47], it is shown that every automorphic system on J^∞ has the form (4.1).

Remark 4.3. In practice we may distinguish the invariants $J^1, \ldots, J^r$ by verifying the independence condition

$$dJ^1 \wedge \cdots \wedge dJ^r \not\equiv 0.$$

on $\mathcal{A}_r$. This may be done symbolically, without the need for explicit formulae for the invariants.

Example 4.4. In this example we obtain the automorphic systems for the pseudo-group (2.3). The differential invariants and their recurrence relations were obtained in Examples 3.7 and 3.9. Since the independent variable $H = H^y = y$ is an invariant, the invariant rank of an automorphic system is bounded by $1 \leq r \leq 2$. Distinguishing the invariants H and J as parameters, the independence condition (3.26) from Example 3.16 implies that when $K \neq 0$, the invariants H, J are independent and automorphic systems of rank 2 have the form

$$\mathcal{A}_2 : \qquad \begin{cases} K = F^1(H, J) \\ L = F^2(H, J) \end{cases} \qquad \Rightarrow \qquad \begin{cases} \dfrac{uu_{xy} - u_x u_y}{u^3} = F^1\left(y, \dfrac{u_y}{u}\right) \\ \dfrac{u_{yy}}{u} = F^2\left(y, \dfrac{u_y}{u}\right) \end{cases} . \tag{4.2}$$

When $K = 0$ we may choose H as a parameter to obtain the rank 1 automorphic systems

$$\mathcal{A}_1 : \qquad \begin{cases} J = F^1(H) \\ K = 0 \\ L = F^2(H) \end{cases} \qquad \Rightarrow \qquad \begin{cases} \dfrac{u_y}{u} = F^1(y) \\ \dfrac{u u_{xy} - u_x u_y}{u^3} = 0 \\ \dfrac{u_{yy}}{u} = F^2(y) \end{cases} . \tag{4.3}$$

The choice of $F^1, \ldots, F^\nu$ in (4.1) may not be arbitrary. Because (4.1) is expressed in terms of differential invariants, applications of syzygies among the invariants will lead to integrability conditions. Consideration of these syzygies leads to a system of differential equations for the functions $F^1, \ldots, F^\nu$ in the automorphic system $\mathcal{A}_r$ that we call the *syzygy system*.

We first discuss syzygy systems for full rank automorphic systems, i.e. $r = p$. Let $\mathcal{S}$ be the set of fundamental syzygies among the generating invariants $\mathcal{I} = \{J^1, \ldots, J^r, K^1, \ldots, K^\nu\}$. Making the chain rule substitutions

$$\mathcal{D}_i = \sum_{k=1}^{r} (\mathcal{D}_i J^k)\, D_{J^k}, \qquad i = 1, \ldots, p, \tag{4.4}$$

we may write the invariant differential operators $\mathcal{D}_i$ appearing in each syzygy in terms of the derivatives D_{J^j}. Without loss of generality, we assume that $\mathcal{D}_i J^j$, $i = 1, \ldots, p$, $j = 1, \ldots, r$, are again functions of the generating invariants $\mathcal{I}$ by increasing the order of prolongation and adding more invariants to $\mathcal{I}$ if necessary (we do not require $\mathcal{I}$ to be minimal). Application of the fundamental syzygies to the system (4.1) results in a system of differential equations for $F^1, \ldots, F^\nu$. This system is called the *syzygy system*.

Remark 4.5. As can be seen from Example 3.16, for any particular symmetry group, the substitution (4.4) may be made symbolically, without explicit formulae for the invariants, using the recurrence relations (3.17).

Example 4.6. The syzygy system associated to the rank 2 automorphic system (4.2) is simply obtained by substituting the functions $K = F^1(H, J)$ and $L = F^2(H, J)$ into the fundamental syzygy (3.28), resulting in the first order partial differential equation

$$F^1 \frac{\partial F^2}{\partial J} = \frac{\partial F^1}{\partial H} + (F^2 - J^2) \frac{\partial F^1}{\partial J} + 3J\, F^1. \tag{4.5}$$

We now address the case when the automorphic systems considered have less than full rank, i.e. $r < p$. In this instance, the substitution (4.4) may introduce new dependencies among the differentiated invariants in addition to the fundamental syzygies and their consequences. We will call these dependencies *restriction syzygies* since they arise from restricting the differential operators to submanifolds (locally) parametrized by $J^1, \ldots, J^r$.

Example 4.7. For the rank 1 automorphic system (4.3), the syzygy (3.28) is trivial. To see this, express the invariant total derivative operators $\mathcal{D}_x$, $\mathcal{D}_y$ in terms of the single operator D_H:

$$\mathcal{D}_x = \mathcal{D}_x H \cdot D_H = 0, \qquad \mathcal{D}_y = \mathcal{D}_y H \cdot D_H = D_H. \tag{4.6}$$

On the other hand, by substitution of (4.6) into the recurrence relations (3.21) we find

$$D_H J = L - J^2, \qquad I_{21} = 0, \qquad I_{12} = 0, \qquad I_{03} = D_H L - JL, \tag{4.7}$$

and so on. Thus, there is a new *restriction syzygy*

$$D_H J = L - J^2$$

among the generating invariants, arising from the restriction of the invariants and invariant differential operators to submanifolds of the form (4.3). It can be seen by inspection that this restriction syzygy is generating. Thus we arrive at the rank 1 syzygy system for the functions $F^1(H), F^2(H)$:

$$\frac{\partial F^1}{\partial H} = F^2 - (F^1)^2.$$

Remark 4.8. We will henceforth refrain from referencing the functions F^i in our examples when it is understood that each invariant K^i is a function $K^i(J^1, \ldots, J^r)$.

Analogous to Theorem 3.14 in the full rank case $r = p$ (where the restriction syzygies are identical to the usual syzygies), the restriction syzygies for $r < p$ are also finitely generated.

Proposition 4.9. Suppose that the Lie pseudo-group $\mathcal{G}$ admits a moving frame. For any choice of distinguished invariants $J^1, \ldots, J^r$, the set of restriction syzygies resulting from substitution of the relations (4.4) into the recurrence relations is finitely generated. A finite generating set of restriction syzygies is called *fundamental restriction syzygies.*

Proof. The full rank case $r = p$ follows from Theorem 3.14. When $r < p$, certain constraints among the differential invariants are imposed, as can be seen in Example 4.4. Writing the differential invariants explicitly in terms of submanifold jet coordinates $(x, u^{(n)})$, these constraints give invariant differential equations that $u = u(x)$ must satisfy. The proposition then follows from the fact that the differential module of differential syzygies restricted to the solution space of an invariant differential equation is finitely generated, [22, Theorem 24]. □

Remark 4.10. Our proof of Proposition 4.9 provides only the existence of a finite generating set of restriction syzygies. A constructive proof would be preferable and useful for more intensive examples than those treated in this paper.

We are now prepared to define the syzygy system for all ranks $r \leq p$.

Definition 4.11. The *syzygy system* $\mathcal{S}_r$ for a rank r automorphic system $\mathcal{A}_r$ is the finite system of differential equations for $F^1, \ldots, F^\nu$ as functions of the invariant parameters $J^1, \ldots, J^r$ obtained by applying to $\mathcal{A}_r$ the fundamental restriction syzygies.

Remark 4.12. It is important to note that the syzygy system does not impose extra conditions on the solutions of the system $\mathcal{A}_r$; $\mathcal{S}_r$ is a collection of integrability conditions on the functions F^j.

Let us now return to the context in which our automorphic systems (4.1) arise as orbits of solutions $u_0(x)$ to a $\mathcal{G}$-invariant differential equation $\Delta = 0$, and discuss how to apply these systems to the problem of finding solutions to $\Delta = 0$.

Starting with a solution $u_0(x)$ to a $\mathcal{G}$-invariant equation $\Delta = 0$, solutions to the $\mathcal{G}$-automorphic system $\mathcal{A}(u_0^{(k)})$ will again satisfy $\Delta = 0$ by invariance. Unfortunately, this observation does not offer obvious practical value for finding solutions to $\Delta = 0$; indeed, if a "seed" solution u_0 is known, one can simply apply the pseudo-group transformations to u_0 and avoid automorphic systems altogether. The preceding construction of syzygy systems suggests an alternative approach: append to the syzygy system the condition $\Delta = 0$. By adding this condition, we ensure that the automorphic systems determined by solving the syzygy system are those generated by solutions to $\Delta = 0$. Note that, by the invariance of $\Delta = 0$, this amounts to adding new relations among the generating invariants; these relations will be called *constraint syzygies*. The constraint syzygies together with the restriction syzygies give a set of differential equations, called the *resolving system*, whose solutions determine automorphic systems generated by solutions of $\Delta = 0$.

Definition 4.13. The rank r *resolving system* $\mathcal{R}_r(\Delta)$ of a differential equation $\Delta = 0$ foliated by $\mathcal{G}$ is the system of differential equations obtained by appending to the syzygy system $\mathcal{S}_r$ the constraint syzygy $\iota(\Delta) = 0$ and its differential consequences.

Example 4.14. We now obtain the rank 2 resolving system for the nonlinear wave equation

$$uu_{xy} - u_x u_y = u^3, \tag{4.8}$$

foliated by the Lie pseudo-group (2.3). First observe that this Lie pseudo-group is a symmetry group of (4.8). Invariantization of (4.8) gives the constraint syzygy

$$K = 1. \tag{4.9}$$

Appending the constraint syzygy to the syzygy system (4.5) yields the resolving system

$$K = 1, \qquad D_J L = 3J. \tag{4.10}$$

Note that there is no rank 1 resolving system because the constraint syzygy (4.9) is not compatible with the dependence condition $K = 0$ from (3.26).

Remark 4.15. The addition of the constraint syzygy may, as usual, be performed symbolically by direct invariantization of the equation $\Delta = 0$ and use of the recurrence relation to write all invariants appearing in $\iota(\Delta)$ in terms of the generating invariants. We assume that the solution space of $\Delta = 0$ lies within the set of regular jets so that the equation may be written as a level set of differential invariants; see [35, Proposition 2.56].

All the ingredients for the group foliation algorithm are now in place.

Algorithm 4.16 (Group foliation)**.** Let $\Delta(x, u^{(n)}) = 0$ be an n-th order differential equation invariant under a Lie pseudo-group $\mathcal{G}$ and suppose that $\mathcal{G}$ admits a moving frame on the solution space of $\Delta = 0$.

• Choose an invariant rank r for which rank r solutions will be sought. Prolong to order $k \geq s$, where s is the order of stabilization of a generic rank r solution, so that the normalized invariants of order at most k form a generating set.

• Choose distinguished invariants $J^1, \ldots, J^r$ among the normalized invariants so that $\mathcal{D}_i J^j$ have order no greater than k. These invariants will be used as independent variables and the remaining normalized invariants $K^1, \ldots, K^\nu$ as dependent variables in the automorphic system

$$\mathcal{A}_r\colon \qquad K^1 = F^1(J^1, \ldots, J^r), \quad \ldots, \quad K^\nu = F^\nu(J^1, \ldots, J^r).$$

• Compute the order r resolving system $\mathcal{R}_r(\Delta)$ by applying the restriction syzygies and the constraint syzygy $\iota(\Delta) = 0$ to $\mathcal{A}_r$.

• Find a solution $F^1(J^1, \ldots, J^r), \ldots, F^\nu(J^1, \ldots, J^r)$ to the resolving system.

• Form an automorphic system $\mathcal{A}_r$ using the resolving system solution and write the invariants in this automorphic system explicitly in terms of $(x, u^{(k)})$. Solutions of this automorphic system will satisfy the original equation $\Delta = 0$.

Example 4.17. We continue Example 4.14. A general solution to the resolving system (4.10) is easily found:

$$K(H, J) = 1, \qquad L(H, J) = \frac{3}{2}J^2 + G(H), \tag{4.11}$$

where $G(H)$ is an arbitrary smooth function. Substituting (4.11) and the explicit formulae (3.13) for the invariants into the automorphic system (4.2) we obtain the system of differential equations

$$\frac{uu_{xy} - u_x u_y}{u^3} = 1, \qquad \frac{u_{yy}}{u} = \frac{3}{2}\left(\frac{u_y}{u}\right)^2 + G(y). \tag{4.12}$$

It is apparent that the method in this instance has been circular; the original equation itself appears in the final automorphic system and the second equation of (4.12) follows from the first by cross-differentiation. This unfortunate outcome will be remedied by the subject of the next section.

Example 4.18. To illustrate the algorithm for non-maximal invariant rank we consider the differential equation

$$uu_{xy} - u_x u_y = 0. \tag{4.13}$$

This equation also admits the symmetry pseudo-group (2.3). Using the same notation as Examples 4.4 and 4.7, (4.13) implies the constraint syzygy $K = 0$. Since $dH \wedge dJ \equiv K\varpi^y \wedge \varpi^x = 0$, the invariants H and J are functionally dependent and the resolving equations in this case are identical to the rank 1 syzygy system already computed in Example 4.7. A solution to the resolving system is

$$J(H) = G(H), \qquad L(H) = G'(H) + G(H)^2, \tag{4.14}$$

where G is an arbitrary smooth function. Substituting (4.14) and the explicit formulae (3.13) for the invariants into the automorphic system (4.3) we obtain the system of differential equations

$$\frac{u_y}{u} = G(y), \qquad \frac{uu_{xy} - u_x u_y}{u^3} = 0, \qquad \frac{u_{yy}}{u} = G'(y) + G(y)^2. \tag{4.15}$$

We do not pursue a solution of (4.15) at present. This will be done by alternative means in Example 4.27 to follow.

In Algorithm 4.16, all steps except for the last may be executed using the symbolic calculus of moving frames. It is only the last step that requires explicit knowledge of the differential invariants and, in the instance of Example 4.17, leads to a dead end in the computation. In keeping with the intent of moving frames, we propose an alternative method for reconstruction of solutions from the resolving system that is completely symbolic, and effective in certain examples — such as Example 4.17 — where the standard reconstruction method fails.

4.2 Reconstruction procedure

The method of moving frames is naturally incorporated into our exposition of the group foliation method. Moving frames are not required *per se* to perform the algorithm, but they facilitate the symbolic construction of the automorphic and resolving systems using only the infinitesimal data of the pseudo-group action and the choice of a cross-section to the pseudo-group orbits. But, when the automorphic system is used to construct a solution to $\Delta = 0$ from a solution of the resolving system, as in (4.12), it becomes necessary to know the explicit formulae for the generating invariants. Also, as Example 4.17 shows, this final step of the group foliation method may result in a problem no easier to solve than the original differential equation.

To address these shortcomings, we replace the explicit automorphic system by a system of *reconstruction equations*. In essence, the reconstruction system makes use of the pseudo-group transformations to map the resolving system solution away from the cross-section, to solutions of $\Delta = 0$. More precisely: a right moving frame ρ will project the jet of an unknown solution along pseudo-group orbits onto the cross-section. This projection is identical to the intersection of the orbit of the solution with the cross-section, and hence characterized as a solution of the resolving system $\mathcal{R}_r$ studied in the previous section. A left moving frame $\overline{\varrho}$ inverts this process, mapping a resolving system solution away from the cross-section and back to solutions of $\Delta = 0$. See Figure 3 for the geometry of this process. We begin by introducing the *pseudo-group jet differentials* of $\mathcal{G}$, which will allow the determination of reconstruction equations in a purely symbolic manner.

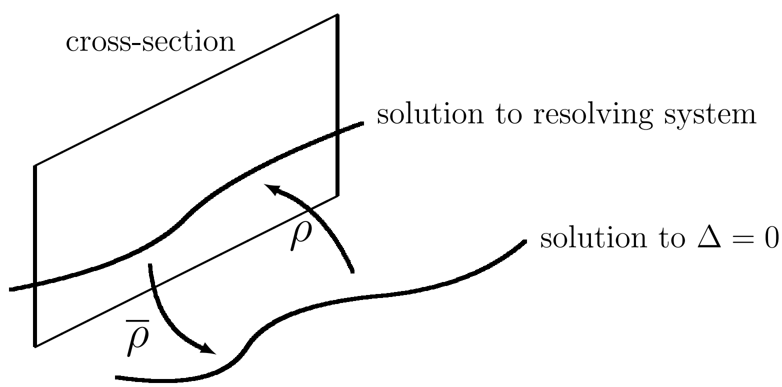


Figure 3: The geometry of reconstruction.

4.2.1 Pseudo-group jet differentials

In this section we introduce the *pseudo-group jet differential expressions* arising simply from taking the exterior derivative of the pseudo-group jets. Pull-back of these pseudo-group jet differentials by the right moving frame results in an expression for the exterior derivatives of the left moving frame components in terms of "known quantities": invariant horizontal differential forms and the right moving frame pull-backs of the right Maurer–Cartan forms, computed using the universal recurrence relation (3.16). Expansion of these exterior derivatives in the invariant horizontal coframe yields differential equations for the the left moving frame; after restriction to a resolving system solution, these differential equations become the *reconstruction equations*.

The pseudo-group jet differential expressions rely on the relation between left and right Maurer–Cartan forms. Recall from Section 2 the right and left zero order Maurer–Cartan forms, respectively, for the full diffeomorphism pseudo-group:

$$\mu^a = dZ^a - \sum_{b=1}^{m} Z^a_{z^b} dz^b, \qquad \overline{\mu}^a = dz^a - \sum_{b=1}^{m} z^a_{Z^b}\, dZ^b.$$

Higher order right and left Maurer–Cartan forms $\mu^a_A = \mathbb{D}^A_Z \mu^a$ and $\overline{\mu}^a_A = \mathbb{D}^A_z \overline{\mu}^a$ are obtained via Lie differentiation with respect to, respectively,

$$\mathbb{D}_{Z^a} = \sum_{b=1}^{m} z^b_{Z^a}\, \mathbb{D}_{z^b} \qquad \text{and} \qquad \mathbb{D}_{z^a} = \sum_{b=1}^{m} Z^b_{z^a}\, \mathbb{D}_{Z^b}.$$

The Maurer–Cartan forms of a Lie pseudo-group $\mathcal{G} \subset \mathcal{D}$ are found by restricting the diffeomorphism pseudo-group Maurer–Cartan forms to the determining equations (2.2) and lifted determining equations (2.11) for $\mathcal{G}$, with the interchange $z \leftrightarrow Z$, $\mu \leftrightarrow \overline{\mu}$ for the left Maurer–Cartan forms.

Using the relation (2.14) between left and right zero order Maurer–Cartan forms we find the following relations among the diffeomorphism pseudo-group jets:

$$dz^a = \sum_{b=1}^{m} (z^a_{Z^b}\, dZ^b - z^a_{Z^b}\, \mu^b). \tag{4.16a}$$

Similar relations among the higher order diffeomorphism pseudo-group jets z^a_A are obtained by Lie differentiation of (4.16a) with respect to $\mathbb{D}_{Z^a}$. For example, we find for the first order pseudo-group jets:

$$dz^a_{Z^c} = \sum_{b=1}^{m} (z^a_{Z^b Z^c}\, dZ^b - z^a_{Z^b Z^c}\, \mu^b - z^a_{Z^b}\, \mu^b_{Z^c}). \tag{4.16b}$$

Definition 4.19. Equations (4.16) and higher order consequences are called *pseudo-group jet differentials* for the diffeomorphism pseudo-group. For a Lie pseudo-group $\mathcal{G} \subset \mathcal{D}$, the pseudo-group jet differentials are obtained by application of the determining system (2.2), with the interchange of z and Z, and lifted determining system (2.11) to (4.16).

Remark 4.20. It will usually be more convenient to work with pseudo-group parameters instead of pseudo-group jets. The distinction is purely computational; we will illustrate both approaches in our running example.

Example 4.21. We now compute the pseudo-group jet differentials for the Lie pseudo-group action (2.3). Applying to (4.16) the determining equations

$$x_Y = 0, \qquad x_U = 0, \qquad y = Y, \qquad u = \frac{U}{x_X},$$

and the lifted determining equations (2.12) for the right Maurer–Cartan forms we obtain

$$\begin{aligned} dx &= x_X\,(dX - \mu^x) \\ dy &= dY \\ du &= \frac{1}{x_X}(dX - \mu^x) - \frac{U x_{XX}}{x_X}(dU + U\mu^x). \end{aligned}$$

Lie differentiation with respect to $\mathbb{D}_X$ gives the higher order relations

$$dx_X = x_{XX}\,(dX - \mu^x) - x_X\,\mu^x_X$$
$$du_X = \frac{-x_{XX}}{x_X^2}(dX - \mu^x) - \frac{1}{x_X}\,\mu^x_X - \frac{U(x_{XXX}x_X - x_{XX}^2)}{x_X^2}(dU + U\mu^x) - \frac{U^2 x_{XX}}{x_X}\,\mu^x,$$

and so on. Writing these jet differentials in terms of the pseudo-group parameters $\bar f = x, \bar f_X = x_X, \bar f_{XX} = x_{XX}, \ldots$ instead offers some simplification:

$$\begin{aligned} d\bar f &= \bar f_X\,(dX - \mu^x), \\ d\bar f_X &= \bar f_{XX}\,dX - \bar f_{XX}\,\mu^x - \bar f_X\,\mu^x_X, \\ d\bar f_{XX} &= \bar f_{XXX}\,dX - \bar f_{XXX}\,\mu^x - 2\bar f_{XX}\,\mu^x_X - \bar f_X\,\mu^x_{XX}, \end{aligned} \tag{4.17}$$

and so on. The pseudo-group jet differentials involving the jets $u, u_X, u_{XX}, \ldots$ may be disregarded since they are expressible in terms of the jet parameters determined by (4.17).

4.2.2 Reconstruction equations

Locally, the right moving frame $\varrho(z^{(\infty)})$ and left moving frame $\overline{\varrho}(z^{(\infty)})$ are completely determined by their pseudo-group jet functions $\rho(z^{(\infty)})$ and $\overline{\rho}(z^{(\infty)})$, respectively. Since the considerations of this section are purely local, we will refer to ρ and $\overline{\rho}$ as the right and left moving frames by an abuse of terminology.

Because the right and left moving frames ρ and $\overline{\rho}$ are related by pseudo-group inversion, the right moving frame pull-back of the "inverse" pseudo-group jets z^a_A produces the left moving frame pull-back of the "regular" pseudo-group jets Z^a_A:

$$\overline{\rho}^*(z,\, Z^{(\infty)}) = \rho^*(Z, z^{(\infty)}).$$

Thus applying the right moving frame pull-back ρ^* to the pseudo-group jet differentials will yield an expression for the differential $d\overline{\rho} = \big(d(\overline{\rho}^* z), d(\overline{\rho}^* z^a_{Z^b}), \ldots\big)$ of the left moving frame:

$$d\overline{\rho} \equiv \sum_{j=1}^{p} P_j(\overline{\rho}, H, I^{(\infty)})\,\varpi^j, \tag{4.18}$$

where $H, I^{(\infty)}$ are the collections of normalized invariants $H^i = \iota(x^i)$, $I^\alpha_J = \iota(u^\alpha_J)$, respectively, and $z = (x, u)$ as usual. The invariants H^i, I^α_J, make their appearance in (4.18) via the normalized Maurer–Cartan forms $\rho^*\mu^a_A$ and the normalized differentials $\rho^* dX^i$, $\rho^* dU^\alpha$. Note that these quantities may all be computed symbolically via the universal recurrence relation (3.16). Giving a general expression for the functions P_j is possible but not necessary for our discussion. To apply the identity (4.18) to the problem of group foliation, we restrict to a particular automorphic system $\mathcal{A}_r$ given by a choice of resolving system solution. First consider the case of full rank $r = p$.

Let $\Delta = 0$ be a $\mathcal{G}$-invariant differential equation, and suppose that a solution to a full rank resolving system $\mathcal{R}_r$ is given, determining the automorphic system $\mathcal{A}_r$. Let $J = \{J^1, \ldots, J^p\}$ be the distinguished independent invariants for the resolving system. Because of independence, invariant horizontal projections of the forms

$$dJ^1, \ldots, dJ^p$$

constitute an invariant horizontal coframe, which may be used in place of $\varpi^1, \ldots, \varpi^p$ in (4.18). Restricted to $\mathcal{A}_r$, (4.18) then yields an explicit system of differential equations for the left moving frame as a function of the distinguished invariants J via projection onto this horizontal coframe:

$$d\overline{\rho} = \sum_{j=1}^{p} Q_j(\overline{\rho}, J)\, dJ^j. \tag{4.19}$$

All invariants $H, I^{(\infty)}$ are expressed as functions of the distinguished invariants via the recurrence relations. The result is a system of first order differential equations that must be satisfied by $\overline{\rho}$:

$$D_{J^j}\, \overline{\rho} = Q_j(\overline{\rho}, J). \tag{4.20}$$

We will refer to (4.19) or (4.20) as *reconstruction equations*.

Theorem 4.22. The reconstruction equations (4.19) are automorphic relative to $\mathcal{G}$.

Proof. Let $\overline{\rho}_1$ and $\overline{\rho}_2$ be two solutions of the reconstruction equations (4.19). Then $S_1 = \overline{\rho}_1 \cdot (H, I)$ and $S_2 = \overline{\rho}_2 \cdot (H, I)$ are p-dimensional submanifolds with same projection onto $\mathcal{K}^\infty$. Since the normalized invariants $(H, I^{(\infty)}) = \iota(x, u^{(\infty)})$ form a complete set of invariants and parametrize $\mathcal{K}^\infty$, the submanifolds S_1 and S_2 have the same signatures, [40, 55]; that is, $(H, I^{(\infty)})|_{S_1} = (H, I^{(\infty)})|_{S_2}$. This implies that there exists a transformation $g \in \mathcal{G}$ such that $g \cdot S_1 = S_2$. By construction of S_1 and S_2, this means that $g^{(\infty)} \cdot \overline{\rho}_1 = \overline{\rho}_2$. □

Remark 4.23. As seen in (3.4), a left moving frame is uniquely determined by its target point. Since the solution to the reconstruction equations (4.20) are expressed in terms of the source coordinates (i.e. coordinates on the cross-section $\mathcal{K}^{(\infty)}$), the solution is not unique. By the automorphic property of the reconstruction solution, if $\overline{\rho}(J)$ is a particular solution, then the general solutions have the form $g^{(\infty)} \cdot \overline{\rho}(J)$ where $g^{(\infty)} \in \mathcal{G}^{(\infty)}$.

Theorem 4.24. The parametrized graph

$$\overline{\rho}(J) \cdot (H(J), I(J)) = (x(J), u(J))$$

is the graph of a solution to the differential equation $\Delta = 0$.

Proof. Let $(H(J), I^{(\infty)}(J))$ be a solution of the resolving system. By definition, this solution must come from the invariantization of some solution $(x, u^{(\infty)}(x))$ to the differential equation $\Delta = 0$. Let $\rho(x)$ be the right moving frame sending $(x, u^{(\infty)}(x))$ onto $(H(J), I^{(\infty)}(J))$. Suppose that $\overline{\rho}$ is a solution to the reconstruction equations. Since $\overline{\rho}$ and ρ^{-1} are both solutions of the reconstruction equations, by the automorphic property there exists $g \in \mathcal{G}$ such that

$$\overline{\rho} = g^{(\infty)} \cdot \rho^{-1}.$$

Since ρ^{-1} maps $(H(J), I^{(\infty)}(J))$ onto the prolonged graph $(x, u^{(\infty)}(x))$, and $g^{(\infty)}$ preserves the property of being a prolonged graph, $\overline{\rho}(J) \cdot (H(J), I^{(\infty)}(J))$ can be identified with $(x, \widetilde{u}^{(\infty)}(x))$ for some function $\widetilde{u}(x)$, which must be a solution of $\Delta = 0$. □

By Theorem 4.24, to construct a solution of $\Delta = 0$, we apply $\overline{\rho}$ to the graph of the resolving system solution:

$$\overline{\rho}(J^1, \ldots, J^p) \cdot \big(H(J^1, \ldots, J^p), I(J^1, \ldots, J^p)\big) = \big(x^1, \ldots, x^p, u^1, \ldots, u^q\big), \tag{4.21}$$

where the normalized invariants $H = \big(\iota(x^1), \ldots, \iota(x^p)\big)$, $I = \big(\iota(u^1), \ldots, \iota(u^q)\big)$ are evaluated on the resolving system solution.

To simplify notation in the following examples, we will use the same notation for the pseudo-group parameters and their right moving frame pull-backs.

Example 4.25. Continuing Example 4.17, we apply the reconstruction approach to obtain solutions to (4.8).We begin by deriving the reconstruction equations (4.19). Taking the right moving frame pull-back of the zero order pseudo-group jet differential from (4.17) yields

$$d\bar{f} = \bar{f}_X \, \varpi^x, \tag{4.22}$$

since, as found in Example 3.9,

$$\rho^*(dX) = 0 \qquad \text{and} \qquad \rho^*(\mu^x) = -\varpi^x.$$

By duality with (3.27) we find

$$\varpi^x \equiv (J^2 - L)\, dH + dJ, \qquad \varpi^y \equiv dH,$$

using the constraint syzygy $K = 1$ and writing $L = L(H, J)$ for our choice of resolving system solution from (4.11). Expressing (4.22) in this new coframe we obtain

$$d\bar{f} = (J^2 - L)\bar{f}_X \, dH + \bar{f}_X \, dJ,$$

which gives the reconstruction equations for $\bar{f}(H, J), \bar{f}_X(H, J)$:

$$D_H \, \bar{f} = (J^2 - L)\bar{f}_X \qquad D_J \bar{f} = \bar{f}_X.$$

These equations determine $\bar{f}, \bar{f}_X$, which are the only parameters needed for reconstruction. Using (4.11) the reconstruction equations may be written more explicitly as

$$D_H \, \bar{f} = -\left(\frac{J^2}{2} + G(H)\right) D_J \bar{f}, \qquad D_J \bar{f} = \bar{f}_X, \tag{4.23}$$

which may be solved by the method of characteristics. Acting on the graph of the resolving system solution in the cross-section (3.11) by the left moving frame determined by the reconstruction yields a solution to the nonlinear wave equation (4.8) given parametrically, in terms of the invariants H and J:

$$x = \bar{f}(H, J), \qquad y = H, \qquad u = \frac{1}{\bar{f}_X(H, J)}.$$

Remark 4.26. The reconstruction result in Example 4.25 was derived in [48] using the machinery of symmetry reduction of exterior differential systems.

We now consider the reconstruction process for non-maximal invariant rank, $r < p$. In this case, invariant horizontal projections of the forms $dJ^1, \ldots, dJ^r$ cannot be used as an invariant coframe in place of the invariant forms ϖ^i. To remedy this situation, we supplement the invariant forms $dJ^1, \ldots, dJ^r$ with $p - r$ forms $\varpi^{j_1}, \ldots, \varpi^{j_{p-r}}$ from the standard invariant horizontal coframe in order to form a full invariant horizontal coframe. Thus the reconstruction equations have the modified form

$$d\overline{\rho} \equiv \sum_{j=1}^{r} Q_j(\overline{\rho}, J^1, \ldots, J^r)\, dJ^j + \sum_{i=1}^{p-r} P_{j_i}(\overline{\rho}, J^1, \ldots, J^r)\, \varpi^{j_i}.$$

We may then use $p - r$ of these equations to express the supplemental differential forms ϖ^{j_i} in terms of the differentials of $p-r$ moving frame components $\overline{\rho}^{a_i} = \rho^*(z^{a_i})$, $i = 1, \ldots, p-r$. Solutions to these non-maximal rank reconstruction equations will then be parametrized by the invariant variables $J^1, \ldots, J^r$ in addition to the components $\overline{\rho}^{a_1}, \ldots, \overline{\rho}^{a_{p-r}}$. The addition of these $p - r$ "free parameters" in the reconstruction transformations is expected; we are attempting to reconstruct the graph of a solution to $\Delta = 0$, a p-dimensional manifold, from the graph of a resolving system solution, a r-dimensional manifold.

Example 4.27. We return to Example 4.18 to illustrate reconstruction for non-maximal rank. Recall that in this example we have the single distinguished invariant H, and the resolving system solution

$$J(H) = G(H)$$
$$L(H) = G'(H) + G(H)^2.$$

We supplement the form dH with ϖ^x so that $\{dH, \varpi^x\}$ is an invariant horizontal coframe. Applying the right moving frame pull-back to the first two pseudo-group jet differentials from (4.17) yields

$$d\bar{f} \equiv \bar{f}_X\, \varpi^x, \qquad d\bar{f}_X \equiv \bar{f}_{XX}\, \varpi^x - \bar{f}_X\, J\, dH. \tag{4.24}$$

The first equation of (4.24) allows us to express the invariant horizontal form ϖ^x in terms of the moving frame components:

$$\varpi^x \equiv d\bar{f}/\bar{f}_X,$$

reducing the second equation of (4.24) to

$$d\bar{f}_X \equiv \frac{\bar{f}_{XX}}{\bar{f}_X}\, d\bar{f} - \bar{f}_X\, J\, dH. \tag{4.25}$$

The component $\bar{f}$ of the moving frame may be taken as an independent variable so that

$$\bar{f}_X = \bar{f}_X(\bar{f}, H), \qquad \bar{f}_{XX} = \bar{f}_{XX}(\bar{f}, H),$$

and hence (4.25) yields differential equations for $\bar{f}_X$:

$$D_{\bar{f}}\bar{f}_X = \frac{\bar{f}_{XX}}{\bar{f}_X}, \qquad D_H\bar{f}_X = -\bar{f}_X\, J.$$

The first equation gives $\bar{f}_{XX}$ in terms of $\bar{f}_X$; solving the second we find

$$\bar{f}_X(\bar{f}, H) = A(\bar{f})\, e^{-\int G(H)\, dH} = \frac{A(\bar{f})}{B(H)},$$

where $A(\bar{f}) \neq 0$, $B(H) > 0$ are arbitrary functions. Hence we find solutions to (4.13), parametrized by $\bar{f}, H$:

$$(x, y, u) = \left(\bar{f}, H, \frac{1}{\bar{f}_X}\right) = \left(\bar{f}, H, \frac{B(H)}{A(\bar{f})}\right).$$

In agreement with our explicit computation of the left moving frame in (3.14), we have $\bar{f} = x$, and conclude that $u(x, y) = B(y)/A(x)$ solves (4.13).

Remark 4.28. Note that due to the automorphic property of the reconstruction equations, solutions are not unique. Acting by a transformation of $\mathcal{G}$ will produce a new reconstruction solution, and hence a new solution to $\Delta = 0$. This freedom of choice in the reconstruction solution can be seen in all of our examples.

Example 4.29. With explicit knowledge of the left moving frame, we can see directly the equivalence of the automorphic system and reconstruction equations. In this example we compare directly the automorphic system (4.12) and reconstruction equations (4.23) for our running example. Taking the exterior derivative of the invariants

$$H = y, \qquad J = \frac{u_y}{u},$$

we obtain

$$dH \equiv dy, \qquad dJ \equiv \left(\frac{u u_{xy} - u_x u_y}{u^2}\right) dx + \left(\frac{u u_{yy} - u_y^2}{u^2}\right) dy,$$

so that by duality

$$D_H = D_y - \left(\frac{u u_{yy} - u_y^2}{u u_{xy} - u_x u_y}\right) D_x, \qquad D_J = \frac{u^2}{u u_{xy} - u_x u_y} D_x.$$

Substituting the values of the left moving frame, $\bar{f} = x$ and $\bar{f}_X = 1/u$, and writing out the reconstruction equations (4.23) explicitly:

$$D_H \bar{f} = \left(\frac{J^2}{2} - G(H)\right) D_J \bar{f}, \qquad D_J \bar{f} = \bar{f}_X$$

we recover the automorphic system (4.2).

5 Further examples

In this section we apply the group foliation method to three other examples. Example 5.1 gives another illustration of the method for an infinite-dimensional symmetry group. Examples 5.3 and 5.4 show how the group foliation method subsumes classical symmetry reduction techniques for finding invariant and partially invariant solutions to differential equations. The symmetry groups appearing in all examples may be obtained via Lie's standard algorithm, [35].

Example 5.1. In this example, we solve the nonlinear Calogero wave equation, [5],

$$u_{xt} + u u_{xx} = F(u_x) \tag{5.1}$$

using the group foliation method. The differential equation (5.1) admits the infinite-dimensional symmetry group

$$X = x + a(t), \qquad T = t, \qquad U = u + a'(t), \tag{5.2}$$

where $a(t)$ is an arbitrary differentiable function of t. The Lie pseudo-group action (5.2) is generated by the vector fields

$$\mathbf{v} = a(t)\frac{\partial}{\partial x} + a'(t)\frac{\partial}{\partial u},$$

whose prolongation is

$$\mathbf{v}^{(\infty)} = a(t)\frac{\partial}{\partial x} + a_t\frac{\partial}{\partial u} + (a_{tt} - u_x\, a_t)\frac{\partial}{\partial u_t} - u_{xx}\, a_t \frac{\partial}{\partial u_{xt}} + (a_{ttt} - u_x\, a_{tt} - 2u_{xt}\, a_t)\frac{\partial}{\partial u_{tt}} + \cdots .$$

The recurrence relations (3.15) for the lifted invariants are

$$\begin{aligned}
&dX = \Omega^x + \mu, \quad dT = \Omega^t,\\
&dU \equiv U_X\, \Omega^x + U_T\, \Omega^t + \mu_T,\\
&dU_X \equiv U_{XX}\, \Omega^x + U_{XT}\, \Omega^t,\\
&dU_T \equiv U_{XT}\, \Omega^x + U_{TT}\, \Omega^t + \mu_{TT} - U_X\, \mu_T,\\
&dU_{XX} \equiv U_{XXX}\, \Omega^x + U_{XXT}\, \Omega^t,\\
&dU_{XT} \equiv U_{XXT}\, \Omega^x + U_{XTT}\, \Omega^t - U_{XX}\mu_T,\\
&dU_{TT} \equiv U_{XTT}\, \Omega^x + U_{TTT}\, \Omega^t + \mu_{TTT} - U_X\, \mu_{TT} - 2U_{XT}\, \mu_T, \quad \ldots .
\end{aligned} \tag{5.3}$$

A cross-section to the pseudo-group orbits is given by

$$X = U_{T^k} = 0, \qquad k \geq 0,$$

which leads to the normalized Maurer–Cartan forms

$$\mu = -\varpi^x, \qquad \mu_T = -I_{10}\, \varpi^x, \qquad \mu_{TT} = -(I_{11} + I_{10}^2)\varpi^x, \qquad \ldots . \tag{5.4}$$

Substituting (5.4) into (5.3) we obtain, up to order 2, the recurrence relations

$$\begin{aligned}
&\mathcal{D}_x I_{10} = I_{20}, && \mathcal{D}_t I_{10} = I_{11},\\
&\mathcal{D}_x I_{20} = I_{30}, && \mathcal{D}_t I_{20} = I_{21},\\
&\mathcal{D}_x I_{11} = I_{21} + I_{10} I_{20}, && \mathcal{D}_t I_{11} = I_{12}.
\end{aligned} \tag{5.5}$$

Eliminating I_{21} from (5.5) we find the syzygy

$$\mathcal{S}\colon \qquad \mathcal{D}_x I_{11} = \mathcal{D}_t I_{20} + I_{10} I_{20}. \tag{5.6}$$

A generating set for the algebra of differential invariants is given by

$$t, \qquad s = I_{10}, \qquad K = I_{11}, \qquad L = I_{20}.$$

For t and s to be independent invariant variables we require that $L \neq 0$ as

$$ds \wedge dt \equiv L\, \varpi^x \wedge \varpi^y.$$

Then, the rank 2 automorphic system is

$$\mathcal{A}_2\colon \qquad K = K(s,t), \qquad L = L(s,t).$$

By the chain rule

$$\mathcal{D}_x = (\mathcal{D}_x t)\, D_t + (\mathcal{D}_x s)\, D_s = L\, D_s,$$
$$\mathcal{D}_t = (\mathcal{D}_t t)\, D_t + (\mathcal{D}_t s)\, D_s = D_t + K\, D_s,$$

and in the variables s, t, the syzygy (5.6) is equivalent to

$$L(K_s - s) = L_t + K L_s. \tag{5.7a}$$

The invariantization of the differential equation (5.1) gives the constraint syzygy

$$K = F(s). \tag{5.7b}$$

Equations (5.7) comprise the resolving system. Substituting (5.7b) into (5.7a) we obtain the first order partial differential equation

$$L_t + F L_s = L(F_s - s) \tag{5.8}$$

for the invariant L. Assuming $F(s) \neq 0$, the solution to (5.8) is

$$L(s,t) = F(s)\, h\biggl(t - \int \frac{ds}{F(s)}\biggr) \exp\biggl[-\int \frac{s}{F(s)} ds\biggr], \tag{5.9}$$

where h is an arbitrary differentiable function. To obtain the solution to the original differential equation (5.1) we solve the reconstruction equation

$$db = \varpi^x + b_T\, dt = \frac{1}{L} ds + \biggl(b_T - \frac{K}{L}\biggr) dt,$$

which implies

$$D_s b = \frac{1}{L} \qquad \text{and} \qquad b_T = D_t b + \frac{K}{L}.$$

Hence,

$$b(s,t) = \int \frac{ds}{L} + a(t) \qquad \text{and} \qquad b_T = -\int \frac{L_t}{L^2} ds + \frac{F(s)}{L} + a'(t),$$

with L given in (5.9). Then, the solutions to (5.1) of invariant rank 2 are

$$(x,t,u) = \overline{\rho} \cdot (0,t,0) = \biggl(\int \frac{ds}{L} + a(t), t, -\int \frac{L_t}{L^2} ds + \frac{F(s)}{L} + a'(t)\biggr), \tag{5.10}$$

where $L(s,t)$ is given by (5.9).

We now assume that $L = 0$, and search for solutions of invariant rank 1. Firstly, the automorphic system is now given by

$$\mathcal{A}_1\colon \qquad s = s(t), \qquad K = K(t), \qquad L = L(t) = 0,$$

and by the chain rule,

$$\mathcal{D}_x = 0, \qquad \mathcal{D}_t = D_t.$$

Then, the recurrence relations (5.5) yield the syzygy

$$D_t s = K,$$

while the constraint syzygy (5.7b) still holds. Thus, the function $s(t)$ is a solution of the ordinary differential equation

$$D_t s = F(s). \tag{5.11}$$

From the pseudo-group jet differentials

$$\begin{aligned} db &= -\mu + b_T\, dt = \varpi^x + b_T\, dt, \\ db_X &= -\mu_T + b_{TT}\, dt = s\, \varpi^x + b_{TT}\, dt, \end{aligned} \tag{5.12}$$

we conclude that

$$\varpi^x = db - b_T\, dt,$$

and so the pseudo-group jets b_T, b_{TT}, $\ldots$ are assumed to be functions of the pseudo-group variable b and the invariant t. From the second equation in (5.12) we deduce that

$$D_b(b_T) = s, \qquad b_{TT} = D_t(b_T) + s\, b_T.$$

Hence

$$b_T(b,t) = b \cdot s(t) + f(t),$$

where $s(t)$ is a solution of (5.11) and $f(t)$ is an arbitrary differentiable function. Finally, the solutions of invariant rank 1 are

$$(x,t,u) = \overline{\rho} \cdot (0,t,0) = (b, t, b \cdot s(t) + f(t)).$$

Remark 5.2. The solution (5.10) also appears in [25]. It can be seen by comparison with this author's computations that the moving frame approach yields the solution in a completely systematic manner and does not require explicit formulae for the invariants s, K and L.

We illustrate in the next two examples the group foliation method for finite dimensional Lie groups. In Example 5.3 we show how the group foliation method subsumes existing algorithms for obtaining invariant, [35], and partially invariant solutions [47]. In the context of finite dimensional Lie groups, the dimension of the automorphic system is bounded between p and $p+r$, where r is the dimension of the Lie group. Let $p+\delta$ be the dimension of the automorphic system, $0 \leq \delta \leq r$. The number δ is called the defect of the solution generating the automorphic system. Invariant solutions have defect $\delta = 0$, while partially invariant solutions satisfy $0 < \delta < r$. By limiting our search to resolving systems of rank $p + \delta - r$, we discover invariant and partially invariant solutions of rank δ.

Finally, Example 5.4 illustrates the use of the group foliation method to reduce the order of an ordinary differential equation. Foliating a second order ordinary differential equation with respect to a two dimensional Lie group, we obtain a resolving system of order zero, i.e. an algebraic equation.

Example 5.3. Consider a system of equations for a transonic gas flow, [47],

$$u_y - v_x = 0, \qquad u\,u_x + v_y = 0. \tag{5.13}$$

To obtain an invariant solution of (5.13) we foliate the equations with respect to the group of dilations

$$X = \lambda\,x, \qquad Y = \lambda\,y, \qquad U = u, \qquad V = v, \tag{5.14}$$

and search for invariant rank 1 solutions of the resolving system. Choosing the cross-section

$$\mathcal{K} = \{y = 1\},$$

a complete set of invariants is given by

$$H = \iota(x), \qquad I_{i,j} = \iota(u_{x^i y^j}), \qquad J_{i,j} = \iota(v_{x^i y^j}).$$

The recurrence relations (3.17) yield

$$dH = \varpi^i - H\,\varpi^y, \tag{5.15}$$

and

$$\begin{aligned} I_{i+1,j} &= \mathcal{D}_x I_{i,j}, \qquad & I_{i,j+1} &= \mathcal{D}_y I_{i,j} - (i+j) I_{i,j}, \\ J_{i+1,j} &= \mathcal{D}_x J_{i,j}, \qquad & J_{i,j+1} &= \mathcal{D}_y J_{i,j} - (i+j) J_{i,j}. \end{aligned}$$

Thus, a generating set of the algebra of differential invariants is given by

$$H, \quad I, \quad J.$$

Modulo the commutator syzygies induced by the commutator relation

$$[\mathcal{D}_y, \mathcal{D}_x] = \mathcal{D}_x,$$

there is no fundamental syzygy.

Searching for order 1 invariant solution, the automorphic system is

$$\mathcal{A}_1\colon \qquad I = I(H), \qquad J = J(H).$$

By the chain rule

$$\mathcal{D}_x = (\mathcal{D}_x H) D_H = D_H, \qquad \mathcal{D}_y = (\mathcal{D}_y H) D_H = -H\,D_H.$$

Thus, the invariantization of the differential equations (5.13) yields

$$H\,D_H I + D_H J = 0, \qquad I\,D_H I - H\,D_H J = 0.$$

Omitting the constant solution, the integration of the resolving system gives

$$I(H) = -H^2, \qquad J(H) = \frac{2}{3} H^3 + C,$$

where C is an arbitrary constant. Implementing the reconstruction step, we obtain the reconstruction equation

$$\varpi^y = \frac{d\overline{\lambda}}{\overline{\lambda}}. \tag{5.16}$$

Viewing $\overline{\lambda}$ as an independent variable, the invariant solution is given by

$$(x, y, u, v) = \overline{\lambda} \cdot (H, 1, I, J) = (H\,\overline{\lambda}, \overline{\lambda}, -H^2, \frac{2}{3}H^3 + C).$$

Since $\overline{\lambda} = y$, $H = x/\overline{\lambda} = x/y$ and the solution invariant under the dilation group (5.14) is

$$u(x, y) = -\left(\frac{x}{y}\right)^2, \qquad v(x, y) = \frac{2}{3}\left(\frac{x}{y}\right)^3 + C.$$

We now obtain a partially invariant solution of (5.3) by foliating (5.13) with respect to

$$X = \lambda\, x, \qquad Y = \lambda\, y, \qquad U = u, \qquad V = v + \epsilon. \tag{5.17}$$

This time, a cross-section is given by

$$\mathcal{K} = \{y = 1,\ v = 0\},$$

and

$$H = \iota(x), \qquad I = \iota(u), \qquad J = \iota(v_x), \qquad K = \iota(v_y),$$

form a generating set of invariants. These invariants admit one fundamental syzygy

$$\mathcal{D}_y J = \mathcal{D}_x K + J. \tag{5.18}$$

Restricting ourself to the rank 1 automorphic system

$$I = I(H), \qquad J = J(H), \qquad K = K(H),$$

the corresponding resolving system is

$$J + H\,D_H I = 0, \qquad K + I\,D_H I = 0, \qquad D_H[(H^2 + I)D_H I] = 0, \tag{5.19}$$

where the first two equations come from the invariantization of (5.13) and the third equation is a consequence of syzygy (5.18). Hence, provided $I(H)$ is a solution of

$$(H^2 + I)D_H I = C,$$

where C is a constant, the invariants J and K are completely determined by (5.19). Implementing the reconstruction step, the first of two reconstruction equations are given by (5.16). From (5.15), which still holds, we conclude that

$$\varpi^x = dH + H\frac{d\overline{\lambda}}{\overline{\lambda}}.$$

Hence, integrating the second reconstruction equation

$$d\overline{\epsilon} = J\,\varpi^x + K\,\varpi^y = -H\,D_H I\,dH - C\,\frac{d\overline{\lambda}}{\overline{\lambda}}$$

we obtain

$$\overline{\epsilon} = -C\ln\overline{\lambda} - \int H D_H I\,dH.$$

This produces the partially invariant solution

$$(x,y,u,v)=(\overline{\lambda},\overline{\epsilon})\cdot(H,1,I,0)=(H\,\overline{\lambda},\overline{\lambda},I(H),-C\ln\overline{\lambda}-\int H\,D_H I\,dH).$$

Since $H=x/y$ and $\overline{\lambda}=y$,

$$u(x,y)=I(x/y),\qquad v(x,y)=-C\ln y-\int(x/y)\,I'(x/y)\,d(x/y).$$

Example 5.4. Consider the nonlinear second order ordinary differential equation

$$x^2u_{xx}=(x\,u_x-u)^2,\qquad x>0. \tag{5.20}$$

The equation (5.20) is invariant under the two dimensional solvable group of transformations

$$X=\lambda\,x,\qquad U=u+\epsilon\,x,\qquad \text{with}\qquad \lambda>0\quad\text{and}\quad \epsilon\in\mathbb{R}. \tag{5.21}$$

In the pseudo-group framework, the determining equations of the Lie group action (5.21) are

$$x\,X_x=X,\qquad X_u=0,\qquad x\,U_x=U-u,\qquad U_u=1,$$

and the infinitesimal determining equations corresponding to an infinitesimal generator $\mathbf{v}=\xi(x,u)\,\partial_x+\phi(x,u)\,\partial_u$ are

$$x\,\xi_x=\xi,\qquad \xi_u=0,\qquad x\,\phi_x=\phi,\qquad \phi_u=0. \tag{5.22}$$

Hence, the general prolonged infinitesimal generator is

$$\mathbf{v}=\xi_x\left(x\frac{\partial}{\partial x}-\sum_{k=1}^{\infty}k\,u_{x^k}\frac{\partial}{\partial u_{x^k}}\right)+\phi_x\left(x\frac{\partial}{\partial u}+\frac{\partial}{\partial u_x}\right),$$

and the order zero lifted recurrence relations are

$$dX=\Omega^x+X\,\mu^x_X,\qquad dU\equiv U_X\,\Omega^x+X\,\mu^u_X. \tag{5.23}$$

Choosing the cross-section $\mathcal{K}^0=\{x=1,\,u=0\}$, the recurrence relations (5.23) yield the normalized Maurer–Cartan forms

$$\mu^x_X=-\varpi^x,\qquad \mu^u_X\equiv -I_1\,\varpi^x.$$

Now, let

$$\mathcal{A}_1:\qquad z=I_1=\iota(u_x)=x\,u_x-u,\qquad v(z)=I_2=\iota(u_{xx})=x^2u_{xx}$$

be the rank 1 automorphic system, which requires that

$$v\neq 0 \tag{5.24}$$

as $dz\equiv v\,\varpi^x$. Since there are no syzygies, the invariantization of the differential equation (5.20) yields the resolving system

$$v(z)=z^2.$$

Hence, the constraint (5.24) is satisfied provided $z \neq 0$. When this is so, $\omega^x = dz/v$ and the reconstruction equations are

$$D_z(x) = \frac{x}{z^2}, \qquad D_z(u) = \frac{u}{z^2} + \frac{1}{z}. \tag{5.25}$$

Solving (5.25) we obtain

$$x(z) = A\, e^{-1/z}, \qquad u(z) = e^{-1/z}\left[\int \frac{e^{1/z}}{z} dz + B\right], \tag{5.26}$$

where A and B are two constants. By construction, the parametric curve (5.26) is a solution of (5.20), to recover the solution in the form $u(x)$ it suffices to express the parameter z as a function of x using the first equation in (5.26):

$$u(x) = -x\left[\int \frac{dx}{x^2(\ln x + A)} + B\right].$$

Remark 5.5. When $\mathcal{G}$ is a (local) Lie group action as in Example 5.4, we can rely on the abstract definition of Lie groups to obtain a simple expression for the reconstruction equations. By Ado's Theorem, [18], every Lie group is locally isomorphic to some linear group $\mathcal{G} \simeq G \subset \mathrm{GL}(k)$ for some $k \in \mathbb{N}$, and a right moving frame is a G-equivariant map $\rho\colon \mathrm{J}^n \to G$ satisfying

$$\rho(g \cdot z^{(n)}) = \rho(z^{(n)}) \cdot g^{-1}.$$

As for Lie pseudo-groups, the corresponding left moving frame is obtained by group inversion $\overline{\rho} = \rho^{-1}$, and the reconstruction equations (4.18) are equivalent to

$$d\overline{\rho} = d\rho^{-1} = -\overline{\rho} \cdot (d\rho \cdot \rho^{-1}) = -\overline{\rho}\, \mu, \tag{5.27}$$

where μ is the moving frame pulled-back Lie algebra valued right-invariant Maurer–Cartan form of G (restricted to a solution of the resolving system). Examples of integrating ordinary differential equations using this point of view can be found in [28, Chapter 6].

6 Normal sub-pseudo-groups

For pseudo-groups admitting normal sub-pseudo-groups it is possible to split the reconstruction procedure into a series of sub-reconstruction steps involving smaller pseudo-groups. This is the moving frame version of Vessiot's observation, [56], that the integration of an automorphic system can be replaced by the integration of a sequence of differential equations automorphic with respect to primitive simple Lie pseudo-groups.

Definition 6.1. A sub-pseudo-group $\mathcal{H} \subseteq \mathcal{G}$ is *normal* if for all $h \in \mathcal{H}$ and $g \in \mathcal{G}$

$$g \circ h \circ g^{-1} \in \mathcal{H} \tag{6.1}$$

whenever the composition is defined.

Let $\mathfrak{h}$ and $\mathfrak{g}$ denote the Lie algebras of $\mathcal{H}$ and $\mathcal{G}$ respectively. Infinitesimally, if $\mathcal{H}$ is a normal sub-pseudo-group of $\mathcal{G}$ then $\mathfrak{h}$ is an ideal of $\mathfrak{g}$:

$$[\mathfrak{h}, \mathfrak{g}] \subseteq \mathfrak{h}.$$

Example 6.2. To illustrate Definition 6.1 we introduce the Lie pseudo-groups

$$\mathcal{H}\colon \qquad X = x, \qquad Y = y + g(x), \qquad U = u + g'(x),$$

and

$$\mathcal{G}\colon \qquad X = f(x), \qquad Y = f'(x)\, y + g(x), \qquad U = u + \frac{f''(x)\, y + g'(x)}{f'(x)}, \tag{6.2}$$

where $f(x) \in \mathcal{D}(\mathbb{R})$ is a local diffeomorphism and $g(x)$ is an arbitrary smooth function. The pseudo-group $\mathcal{H}$ is a sub-pseudo-group of $\mathcal{G}$ obtained by setting $f = \mathbb{1}$ to be the identity map in (6.2). To verify (6.1), let

$$g \cdot (x,y,u) = \left(f(x), f'(x)\, y + g(x), u + \frac{f''(x)\, y + g'(x)}{f'(x)} \right) \quad \in \quad \mathcal{G}$$

and

$$g^{-1} \cdot (X,Y,U) = \left(F(X), F'(X)\, Y + G(X), U + \frac{F''(X)\, Y + G'(X)}{F'(X)} \right),$$

where $F(X) = f^{-1}(X)$ and $G(X) = -g(F(X))/f'(F(X))$. If

$$h \cdot (x,y,u) = \big(x, y + h(x), u + h'(x)\big) \quad \in \quad \mathcal{H},$$

a direct computation shows thats

$$g \circ h \circ g^{-1} \cdot (X,Y,U) = (X, Y + H(X), U + H'(X)) \quad \in \quad \mathcal{H},$$

with $H(X) = f'(F(X)) \cdot h(F(X))$. Infinitesimally, the Lie algebras of $\mathcal{G}$ and $\mathcal{H}$ are spanned by

$$\begin{aligned} \mathfrak{g} &= \operatorname{span}\left\{ \mathbf{v}_a = a(x)\frac{\partial}{\partial x} + y\, a'(x)\frac{\partial}{\partial y} + y\, a''(x)\frac{\partial}{\partial u},\ \mathbf{w}_b = b(x)\frac{\partial}{\partial y} + b'(x)\frac{\partial}{\partial u} \right\}, \\ \mathfrak{h} &= \operatorname{span}\left\{ \mathbf{w}_b = b(x)\frac{\partial}{\partial y} + b'(x)\frac{\partial}{\partial u} \right\}, \end{aligned} \tag{6.3}$$

where $a(x)$ and $b(x)$ are arbitrary smooth functions. Computing the basic commutators

$$[\mathbf{v}_a,\, \mathbf{v}_b] = \mathbf{v}_{a\, b' - b\, a'}, \qquad [\mathbf{w}_a,\, \mathbf{w}_b] = 0, \qquad [\mathbf{v}_a,\, \mathbf{w}_b] = \mathbf{w}_{a\, b' - b\, a'},$$

we see that $\mathfrak{h}$ is an abelian ideal of $\mathfrak{g}$.

Given a normal Lie sub-pseudo-group $\mathcal{H}$ of $\mathcal{G}$, the definition of the quotient Lie pseudo-group of $\mathcal{G}$ by $\mathcal{H}$ is based on the notion of *invariant admissible fibration* introduced by Rodrigues in [49]. We now recast the main definitions of [49] at the pseudo-group level. Given a fibered manifold $\pi\colon M \to N$ and a Lie pseudo-group $\mathcal{G}$ acting on M, a local diffeomorphism $g \in \mathcal{G}$ is said to be projectable by π if there exists a local diffeomorphism $\varphi \in \mathcal{D}(N)$ such that $\pi \circ g = \varphi \circ \pi$. We denote by $\widetilde{\pi}(g) = \varphi$ the map that sends the projectable diffeomorphism g to its projection φ. The fibration $\pi\colon M \to N$ is said to be $\mathcal{G}$-*invariant* if every pseudo-group transformation $g \in \mathcal{G}$ is projectable.

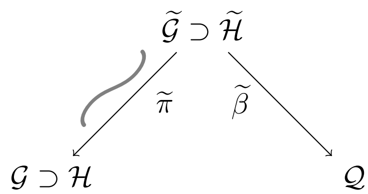
Recall from Definition 2.1 that the map $\pi^{n+k}_n\colon \mathcal{G}^{(n+k)} \to \mathcal{G}^{(n)}$ denotes the standard pseudo-group jet projection.

Definition 6.3. A $\mathcal{G}$-invariant fibration $\pi\colon M \to N$ is called *$\mathcal{G}$-admissible* if there are integers n_0 and k_0 such that $(\ker\widetilde{\pi})^{(n)} \cap \mathcal{G}^{(n)}$ and $\pi^{n+k}_n((\ker\widetilde{\pi})^{(n+k)} \cap \mathcal{G}^{(n+k)})$ are sub-bundles of the pseudo-group jet bundle $\mathcal{G}^{(n)}$ for $n \geq n_0$ and $k \geq k_0$.

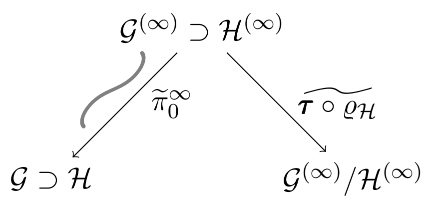
Definition 6.4. Let $\mathcal{G}$ and $\mathcal{H}$ be two Lie pseudo-groups acting on M and N, respectively. A *homomorphism* of $\mathcal{G}$ onto $\mathcal{H}$ is a fibration $\pi\colon M \to N$ which is $\mathcal{G}$-admissible and such that $\widetilde{\pi}(\mathcal{G}) = \mathcal{H}$. If the kernel $\ker\widetilde{\pi}$ is trivial, then $\widetilde{\pi}$ is said to be an *isomorphism* of $\mathcal{G}$ onto $\mathcal{H}$.

Definition 6.5. Let $\mathcal{H}$ be a normal sub-Lie pseudo-group of $\mathcal{G}$. A Lie pseudo-group $\mathcal{Q}$ is a quotient of $\mathcal{G}$ by $\mathcal{H}$ if there exists Lie pseudo-groups $\widetilde{\mathcal{G}}$ and $\widetilde{\mathcal{H}} \subset \widetilde{\mathcal{G}}$, an isomorphism $\widetilde{\pi}\colon \widetilde{\mathcal{G}} \to \mathcal{G}$ such that $\widetilde{\pi}(\widetilde{\mathcal{H}}) = \mathcal{H}$ and a homomorphism $\widetilde{\beta}\colon \widetilde{\mathcal{G}} \to \mathcal{Q}$ whose kernel is $\widetilde{\mathcal{H}}$.

Pictorially, we have



The last three definitions naturally fit within the moving frame framework. Given a Lie pseudo-group $\mathcal{G}$ acting on M and a normal Lie sub-pseudo-group $\mathcal{H} \subset \mathcal{G}$, consider their isomorphic prolongations $\mathcal{G}^{(\infty)}$ and $\mathcal{H}^{(\infty)}$ obtained by considering their prolonged action on the set of regular jets $\mathcal{V}^\infty \subset \mathrm{J}^\infty$ of $\mathcal{G}^{(\infty)}$. Let $\mathcal{K}_\mathcal{H}$ be a cross-section to the $\mathcal{H}^{(\infty)}$-orbits and let $\varrho_\mathcal{H}$ be the corresponding right moving frame. Then, the projection $\boldsymbol{\tau} \circ \varrho_\mathcal{H}\colon \mathcal{V}^\infty \to \mathcal{K}_\mathcal{H}$ onto the cross-section $\mathcal{K}_\mathcal{H}$ is a $\mathcal{G}^{(\infty)}$-invariant admissible fibration of $\mathcal{V}^\infty$. The quotient of $\mathcal{G}$ by $\mathcal{H}$ can then be identified with the projected action of $\mathcal{G}^{(\infty)}$ onto $\mathcal{K}_\mathcal{H}$ which we will write as $\mathcal{G}^{(\infty)}/\mathcal{H}^{(\infty)}$:



Since $\mathcal{K}_\mathcal{H}$ can be identified with the space of $\mathcal{H}$-invariants, the quotient pseudo-group $\mathcal{G}^{(\infty)}/\mathcal{H}^{(\infty)}$ has a well-defined action on the space of $\mathcal{H}$-invariants. Finally, we note that the quotient pseudo-group $\mathcal{G}^{(\infty)}/\mathcal{H}^{(\infty)}$ is isomorphic, [51], to the sub-pseudo-group of transformations of $\mathcal{G}$ that keep the cross-section $\mathcal{K}_\mathcal{H}$ invariant:

$$\mathcal{G}/\mathcal{H} = \{g \in \mathcal{G} \mid g^{(\infty)} \cdot \mathcal{K}_\mathcal{H} \in \mathcal{K}_\mathcal{H}\} \subset \mathcal{G}.$$

Example 6.6. Continuing Example 6.2, we now implement the moving frame method for the normal sub-pseudo-group $\mathcal{H}$. Up to order 2, the lifted invariants are

$$X = x, \qquad Y = y + g(x), \qquad U = u + g_x, \qquad U_X = u_x + g_{xx} - g_x\, u_y, \qquad U_Y = u_y,$$
$$U_{XX} = u_{xx} + g_{xxx} - g_{xx}\, u_y - 2g_x\, u_{xy} + g_x^2\, u_{yy}, \quad U_{XY} = u_{xy} - g_x\, u_{yy}, \quad U_{YY} = u_{yy}.$$

Choosing the cross-section

$$\mathcal{K}^\infty_{\mathcal{H}} = \{y = u_{x^k} = 0,\ k \geq 0\} \tag{6.4}$$

we find, up to order 3, the invariants

$$\begin{gathered} X = x, \qquad I_{01} = u_y, \qquad I_{11} = u_{xy} + u\,u_{yy}, \qquad I_{02} = u_{yy}, \\ I_{21} = u_{xxy} + (u_x + u\,u_y)u_{yy} + 2u\,u_{xyy} + u^2 u_{yyy}, \qquad I_{12} = u_{xyy} + u\,u_{yyy}, \qquad I_{03} = u_{yyy}. \end{gathered} \tag{6.5}$$

We now introduce the quotient pseudo-group $\mathcal{G}^{(\infty)}/\mathcal{H}^{(\infty)}$ acting on the $\mathcal{H}$-invariants (6.5):

$$\begin{gathered} X = f(x), \qquad J_{01} = \frac{I_{01}}{f_x} + \frac{f_{xx}}{f_x^2}, \qquad J_{11} = \frac{I_{11}}{f_x^2} + \frac{f_{xxx} - f_{xx}\,I_{01}}{f_x^3} - 2\frac{f_{xx}^2}{f_x^4}, \\ J_{02} = \frac{I_{02}}{f_x^2}, \qquad J_{12} = \frac{f_x\,I_{12} - 2f_{xx}\,I_{02}}{f_x^4}, \qquad J_{03} = \frac{I_{03}}{f_x^3}, \qquad \ldots . \end{gathered} \tag{6.6}$$

In the above action formulas, $J_{i,j}$ denotes the image of the invariant $I_{i,j}$. This pseudo-group is the prolongation to the $\mathcal{H}$-invariants of the sub-pseudo-group

$$\mathcal{G}/\mathcal{H}\colon \qquad X = f(x), \qquad Y = f'(x)\,y, \qquad U = u + \frac{f''(x)\,y}{f'(x)} \tag{6.7}$$

of $\mathcal{G}$ that fixes the cross-section (6.4). The pseudo-group (6.7) originally appeared in [31], where Medolaghi systematically studies isomorphic representations of the diffeomorphism pseudo-group $\mathcal{D}(\mathbb{R})$. This pseudo-group was also used by Vessiot, [56], in his work on automorphic systems.

Given a differential equation $\Delta = 0$ with symmetry pseudo-group $\mathcal{G}$, if we assume that $\mathcal{H} \subset \mathcal{G}$ is a normal sub-pseudo-group, then it is possible to apply the group foliation procedure iteratively. First, we foliate the solution space of the differential equation with respect to the normal sub-pseudo-group $\mathcal{H}$ and project solutions onto the cross-section $\mathcal{K}_{\mathcal{H}}$ defining a moving frame for $\mathcal{H}$. Let $\mathcal{A}^{\mathcal{H}}_r$ and $\mathcal{R}^{\mathcal{H}}_r$ be the corresponding automorphic and resolving systems.

Proposition 6.7. The resolving system $\mathcal{R}^{\mathcal{H}}_r$ is invariant under the quotient pseudo-group[1] $\mathcal{G}/\mathcal{H}$.

Proof. Since syzygies among differential invariants are invariant under the diffeormorphism pseudo-group $\mathcal{D}(M)$, it follows that $\mathcal{R}^{\mathcal{H}}_r$ in invariant under the quotient pseudo-group $\mathcal{G}/\mathcal{H}$ as the differential equation $\Delta = 0$ is $\mathcal{G}$-invariant. □

The invariance of resolving system $\mathcal{R}^{\mathcal{H}}_r$ under $\mathcal{G}/\mathcal{H}$ permits us to foliate the solution space of $\mathcal{R}^{\mathcal{H}}_r$ with respect to the quotient pseudo-group $\mathcal{G}/\mathcal{H}$. The result is the same as foliating the differential equation $\Delta = 0$ by the full symmetry pseudo-group $\mathcal{G}$.

Assuming a solution to the resolving system $\mathcal{R}^{\mathcal{G}}_{\widetilde{r}}$ is known, Figure 4 shows that the reconstruction operation splits in two steps. First, we can solve the reconstruction equations for the quotient pseudo-group $\mathcal{G}/\mathcal{H}$ to obtain a solution to the resolving

[1] To be more accurate, we should write $\mathcal{G}^{(\infty)}/\mathcal{H}^{(\infty)}$ instead of $\mathcal{G}/\mathcal{H}$, but since the two pseudo-groups are isomorphic, we, from now on, use the latter to simplify the notation.

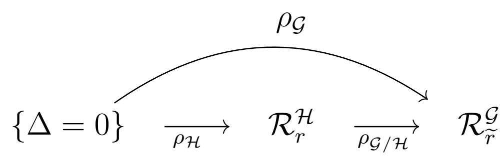


Figure 4: Iterative group foliation.

system $\mathcal{R}^{\mathcal{H}}_r$. Given a solution, solving the reconstruction equations for the normal pseudo-group $\mathcal{H}$ yields a solution to the original differential equation. At the level of moving frames, this reflects the fact that the left $\mathcal{G}$-equivariant moving frame is equivalent to the composition

$$\overline{\rho}_{\mathcal{G}} = \overline{\rho}_{\mathcal{H}} \cdot \overline{\rho}_{\mathcal{G}/\mathcal{H}}.$$

In general, the reconstruction procedure can split in many steps. Given a Lie pseudo-group $\mathcal{G}$, let $\mathcal{G}_1 \subsetneq \mathcal{G}$ be a proper maximal normal sub-Lie pseudo-group of $\mathcal{G}$. Similarly, let $\mathcal{G}_2 \subsetneq \mathcal{G}_1$ be a proper maximal normal sub-Lie pseudo-group of $\mathcal{G}_1$. Repeating the procedure, assume it is possible to obtain a finite a chain of sub-Lie pseudo-groups

$$\{\mathbb{1}\} = \mathcal{G}_{\ell+1} \subsetneq \mathcal{G}_\ell \subsetneq \mathcal{G}_{\ell-1} \subsetneq \cdots \subsetneq \mathcal{G}_1 \subsetneq \mathcal{G}_0 = \mathcal{G},$$

such that for each $k = 0, \ldots, \ell$, the quotient $\overline{\mathcal{G}}_k = \mathcal{G}_k/\mathcal{G}_{k+1}$ is a simple pseudo-group. Then, the group foliation method reduces to solving the resolving system $\mathcal{R}^{\mathcal{G}}_r$ followed by a series of reconstruction steps for the simple sub-Lie pseudo-groups $\overline{\mathcal{G}}_\ell, \ldots, \overline{\mathcal{G}}_0$. According to Cartan, [6], each sub-reconstruction step will only require the integration of either ordinary differential equations or linear partial differential equations involving no more than one arbitrary function.

Example 6.8. To illustrate the iterative reconstruction procedure, we foliate the differential equation

$$u_{xyy} + u\, u_{yyy} + 2u_y\, u_{yy} = 0 \tag{6.8}$$

with respect to the symmetry pseudo-group (6.2). The moving frame construction for this Lie pseudo-group can be found in [42, 43]. Choosing the cross-section

$$\mathcal{K}^\infty_{\mathcal{G}} = \{x = y = u_{x^k} = u_{yx^k} = 0,\ u_{yy} = 1 \ :\ k \geq 0\}, \tag{6.9}$$

and letting $I_{ij} = \iota(u_{x^iy^j})$ and $\mu_k = \iota(a_k)$, $\nu_k = \iota(b_k)$, where $a(x)$, $b(x)$ are the arbitrary functions occurring in the infinitesimal generators (6.3), the normalized Maurer–Cartan forms of order ≤ 2 are

$$\mu \equiv -\varpi^x, \qquad \mu_X \equiv \frac{1}{2}(I_{12}\,\varpi^x + I_{03}\,\varpi^y), \qquad \mu_{XX} \equiv \nu \equiv -\varpi^y, \qquad \nu_X \equiv \nu_{XX} \equiv 0. \tag{6.10}$$

The recurrence relations for the third order normalized invariants are

$$\begin{aligned} \mathcal{D}_x I_{12} &= I_{22} - \frac{3}{2}I_{12}^2, & \mathcal{D}_y I_{12} &= I_{13} - \frac{3}{2}I_{12}I_{03} + 2, \\ \mathcal{D}_x I_{03} &= I_{13} - \frac{3}{2}I_{12}I_{03}, & \mathcal{D}_y I_{03} &= I_{04} - \frac{3}{2}I_{03}^2, \end{aligned} \tag{6.11}$$

while the fundamental syzygy, modulo the commutator relation

$$[\mathcal{D}_x, \mathcal{D}_y] = \frac{I_{03}}{2}\mathcal{D}_x - \frac{I_{12}}{2}\mathcal{D}_y, \tag{6.12}$$

is given by

$$\mathcal{D}_x I_{03} - \mathcal{D}_y I_{12} = -2. \tag{6.13}$$

Implementing the group foliation algorithm, we let the third order normalized invariants

$$s = \iota(u_{xyy}), \qquad t = \iota(u_{yyy}),$$

play the role of the independent invariants, and let the fourth order normalized invariants

$$I_{22}(s,t) = \iota(u_{x^2y^2}), \qquad I_{13}(s,t) = \iota(u_{xy^3}), \qquad I_{04}(s,t) = \iota(u_{y^4}) \tag{6.14}$$

be the dependent invariants so that (6.14) forms the automorphic system. Then, the invariantization of (6.8) yields

$$s = 0. \tag{6.15}$$

Hence, the automorphic system (6.14) will produce invariant rank 1 solutions. Taking into account (6.15) the fundamental syzygy (6.13) yields

$$\mathcal{D}_x t = -2. \tag{6.16}$$

By the chain rule

$$\mathcal{D}_x = (\mathcal{D}_x s) D_s + (\mathcal{D}_x t) D_t = -2 D_t, \qquad \mathcal{D}_y = (\mathcal{D}_y s) D_s + (\mathcal{D}_y t) D_t = I\, D_t,$$

where $I(t) = \mathcal{D}_y t = I_{04} - 3t^2/2$ by the recurrence relations (6.11). Differentiating the fundamental syzygy (6.16) with respect to $\mathcal{D}_y$ and using the commutation relation (6.12) we deduce the differential equation

$$2 D_t I = t \qquad \text{or} \qquad D_t I_{04} = \frac{7}{2} t. \tag{6.17}$$

On the other hand, the recurrence relations (6.11) imply that

$$I_{22} = 0 \qquad \text{and} \qquad I_{13} = -2. \tag{6.18}$$

In summary, the equations (6.15), (6.17), and (6.18) form the resolving system. Integrating (6.17) we obtain

$$I(t) = \frac{a^2 + t^2}{4} \qquad \text{or} \qquad I_{04}(t) = \frac{a^2 + 7t^2}{4},$$

where a is a constant of integration.

We are now ready to implement the reconstruction procedure. Based on Examples 6.2 and 6.6, we first implement the reconstruction procedure for the quotient pseudo-group (6.6). Using the Maurer–Cartan normalizations (6.10), the reconstruction equations (4.18) for the quotient action are, up to order 2,

$$d\bar{f} = \bar{f}_X\, \varpi^x, \qquad d\bar{f}_X = \bar{f}_{XX}\, \varpi^x - \frac{t}{2} \bar{f}_X\, \varpi^y, \qquad d\bar{f}_{XX} = \bar{f}_{XXX}\, \varpi^x + (\bar{f}_X - t\, \bar{f}_{XX}) \varpi^y. \tag{6.19}$$

From the first equation, we have that

$$\varpi^x = d\bar{f}/\bar{f}_X. \tag{6.20}$$

On the other hand, from the equality

$$dt = (\mathcal{D}_x t)\varpi^x + (\mathcal{D}_y t)\varpi^y = -2\,\varpi^x + I\,\varpi^y,$$

we have that

$$\varpi^y = \frac{1}{I}\left(dt + 2\frac{d\bar{f}}{\bar{f}_X}\right). \tag{6.21}$$

The second and third equations of (6.19) then reduce to

$$D_t \bar{f}_X = -\frac{t\,\bar{f}_X}{2I}, \qquad D_{\bar{f}} \bar{f}_X = \frac{\bar{f}_{XX}}{\bar{f}_X} - \frac{t}{I}, \tag{6.22a}$$

$$D_t \bar{f}_{XX} = \frac{\bar{f}_X - t\,\bar{f}_{XX}}{I}, \qquad D_{\bar{f}} \bar{f}_{XX} = \frac{\bar{f}_{XXX}}{\bar{f}_X} + \frac{2(\bar{f}_X - t\bar{f}_{XX})}{I\bar{f}_X}. \tag{6.22b}$$

From (6.22a) we deduce that

$$\bar{f}_X(t,\bar{f}) = \frac{F(\bar{f})}{a^2+t^2}, \qquad \bar{f}_{XX}(t,\bar{f}) = \frac{F(\bar{f})(F'(\bar{f})-4t)}{(a^2+t^2)^2}, \tag{6.23}$$

where $F(\bar{f}) \neq 0$ is an arbitrary nonzero smooth function. At the next order, substituting (6.23) into the first equation of (6.22b) we see that the equation is identically satisfied while the second equation of (6.22b) defines $\bar{f}_{XXX}$. Continuing the computation at higher order we obtain the expressions for $\bar{f}_{X^k}$, $k \geq 4$, giving the left moving frame $\overline{\rho}_{\mathcal{G}/\mathcal{H}}$.

The next step in the iterative reconstruction procedure is to construct the left moving frame $\overline{\rho}_{\mathcal{H}}$. To obtain the reconstruction equations for the pseudo-group jets $\bar{g}$, $\bar{g}_X$, …, we first compute (4.16) for the full symmetry pseudo-group (6.2) and then restrict the equations to the cross-section (6.4). This yields

$$d\bar{g} = \bar{g}_X\,\varpi^x + \bar{f}_X\,\varpi^y, \qquad d\bar{g}_X = \bar{g}_{XX}\varpi^x + \left(\bar{f}_{XX} - \frac{t\,\bar{g}_X}{2}\right)\varpi^y, \qquad \ldots.$$

Using (6.20), (6.21), and (6.23), we obtain

$$D_t\bar{g} = \frac{4F(\bar{f})}{(t^2+a^2)^2}, \qquad \frac{\bar{g}_X}{\bar{f}_X} = D_{\bar{f}}\bar{g} - \frac{2}{I}.$$

Integrating the first equation, we find that

$$\bar{g} = 4F(\bar{f})\left(\frac{t}{2a^2(t^2+a^2)} + \frac{1}{2a^3}\arctan\frac{t}{a}\right) + G(\bar{f}),$$

$$\frac{\bar{g}_X}{\bar{f}_X} = 4F'(\bar{f})\left(\frac{t}{2a^2(t^2+a^2)} + \frac{1}{2a^3}\arctan\frac{t}{a}\right) + G'(\bar{f}) - \frac{8}{t^2+a^2},$$

where $G(\bar{f})$ is an arbitrary smooth function. Thus, a parametrized solution to the partial differential equation (6.8) is given by

$$(x,y,u) = \overline{\rho}_{\mathcal{G}} = \overline{\rho}_{\mathcal{H}} \cdot \overline{\rho}_{\mathcal{G}/\mathcal{H}} \cdot (0,0,0) = \left(\bar{f}, \bar{g}, \frac{\bar{g}_X}{\bar{f}_X}\right).$$

Conclusion

Using the machinery of equivariant moving frames, we have attempted to provide a unified and computationally clear approach to group foliation. The newness and broad applicability of moving frame theory brings fresh insight to old algorithms and as such, many unexplored directions present themselves. We list here several possibilities for further research.

(a) One of the most obvious applications of group foliation is to the solution of differential equations. There are many physically interesting equations that may be particularly amenable to our version of group foliation because of the complexity of their symmetry pseudo-groups. Four particularly interesting examples are: the Davey–Stewartson equations, [7], the Infeld–Rowlands equation, [12], the potential Kadomstev–Petviashvili equation, [11], and the Calabi–Yau equation for Kähler–Einstein metrics, [57].

(b) The results and algorithms presented in the present paper relied on the construction of a moving frame. By appropriately adapting the exposition, it is possible to encompass the situation where only a partial moving frame, [46, 55], exists on the solution space of a differential equation. For example, when foliating (4.8) with respect to its full symmetry group

$$X = f(x), \qquad Y = g(y), \qquad U = \frac{u}{f'(x)\, g'(y)}, \tag{6.24}$$

no differential invariants exist on the solution space of the differential equation and only a partial moving frame can be constructed. In this case, this is a reflection of the automorphic property of (4.8) with respect to the pseudo-group action (6.24). A more detailed investigation of the group foliation method in the context of partial moving frames could be of interest.

(c) The systematic construction of Bäcklund transformations using symmetry reduction of exterior differential systems introduced by Anderson and Fels applies only to Lie groups. The group foliation/inductive moving frames approach outlined in Section 6 does not have this limitation. It would be worthwhile to pursue the possibility of constructing new Bäcklund transformations by realizing systems of interest as resolving systems for infinite-dimensional Lie pseudo-groups; these ideas are also most likely closely related to the reduction methods for infinite dimensional Lie pseudo-groups introduced by Pohjanpelto, [48]. The investigation of non-maximal rank resolving systems could also produce interesting examples. Finally, the group foliation algorithm in conjunction with inductive moving frames may provide a means for constructing coverings of differential equations, [16, 20].

(d) Through the use of joint moving frames and joint invariants, [37, 38], the moving frames approach to group foliation may be adapted to finite difference equations. This adaptation is the subject of a work in progress, [53]. Investigating the possibility of discrete group foliation as a numerical method for solving differential equations could be fruitful.

It may also be worthwhile to pursue the construction of Bäcklund transformations for finite difference or differential-difference equations and compare these results with similar notions from discrete differential geometry and integrable systems, [4, 10].

(e) In the case of group foliation by finite-dimensional Lie group actions, non-maximal rank resolving systems correspond to what are called partially invariant solutions, [47]. The question of when a partially invariant solution is *irreducible*, i.e. not obtainable as an invariant or partially invariant solution for a subgroup, has been studied by Ondich, [45]. This allows for an extension of the classification of group invariant solutions, [35], to partially invariant solutions. It may be interesting to investigate the extension of such a classification and the notion of irreducibility in the context of group foliation.

(f) Invariant submanifold flows find applications in a diversity of fields such as control theory, [33], elasticity theory, [24], and computer vision, [8, 50], and it is possible that the idea of invariant flow reconstruction presented in [52] may provide insight in some of these areas. Theoretical application of invariant flow reconstruction is also worth exploring. For example, Mansfield and van der Kamp, [29], have studied the question of when the integrability (in the sense of possessing infinitely many symmetries) of a differential invariant signature flow "lifts" to integrability of the flow itself. We suspect that their results could be reinterpreted within our framework.

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