# Group foliation of finite difference equations


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**Abstract**

Using the theory of equivariant moving frames, a group foliation method for invariant finite difference equations is developed. This method is analogous to the group foliation of differential equations and uses the symmetry group of the equation to decompose the solution process into two steps, called resolving and reconstruction. Our constructions are performed algorithmically and symbolically by making use of discrete recurrence relations among joint invariants. Applications to invariant finite difference equations that approximate differential equations are given.


# 1 Introduction

First introduced by Sophus Lie, [21], and further developed by Ernest Vessiot, [40], the method of group foliation, also called group splitting, or group stratification, is a general procedure for obtaining solutions of differential equations admitting a symmetry group. Modern treatments of this method appear in the book of Ovsiannikov, [33], and the work of Martina, Nutku, Sheftel and Winternitz, [25, 27]. Recently, a formulation based on exterior differential systems was developed by Anderson, Fels and Pohjanpelto, [1,9,34]. Another formulation based on the theory of equivariant moving frames was proposed in [23] for finite-dimensional symmetry groups and extended to infinite-dimensional symmetry groups in [38]. In the present work, we adapt the constructions introduced in [23, 38] to finite difference equations.

Given a strongly $G$-invariant finite difference equation (see Definition 2.5), the group foliation method uses the foliation of the solution space of the equation by its symmetry group to split the search for solutions into two steps: a *resolving step* and a *reconstruction step*. In the resolving step, the solution space of the equation is projected onto the leaves of the foliation, where, under typical regularity assumptions, the leaves of the foliation are parameterized by joint invariants (also called finite difference invariants). In applications, the projection is obtained by solving a system of equations consisting of the original equation written in terms of joint invariants together with the integrability conditions originating from the syzygies among the invariants. In Vessiot's terminology, these equations form the *resolving system*. As Ovsiannikov observed in [33], the resolving system may be easier to solve than the original equation since the symmetry group has reduced the size of the solution space. Given a solution to the resolving system, the reconstruction step then consists of solving a system of first order finite difference equations for the left moving frame of $G$, called the *reconstruction equations*. Solutions to the original finite difference equation are then obtained by acting on the solution of the resolving system by solutions of the reconstruction equations.

Using the theory of discrete equivariant moving frames, the resolving and reconstruction steps described above can be performed algorithmically and symbolically. After reviewing the concepts of finite difference equations and symmetry in Section 2, the basic moving frame constructions are introduced in Section 3. Since our emphasis is geared towards developing the group foliation method, we refer to [24] for some of the more subtle theoretical justifications of the discrete moving frame method. We note that our notation and terminology differs slightly from that used in [24].

In the differential setting, one of the fundamental results of the equivariant moving frame method is the derivation of recurrence relations relating the normalized differential invariants and their exterior derivatives. In Section 4, a discrete version of the recurrence relations is introduced, relating normalized joint invariants and their shifts. In analogy with the continuous theory, the discrete recurrence relations reveal the structure of the algebra of joint invariants, and these recurrence relations can be computed symbolically without knowing the expressions for the joint invariants, requiring only the expressions for the group action and the choice of a cross-section defining a moving frame. As a result, the group foliation algorithm introduced in Section 6 is completely symbolic in the sense that it does not require coordinate expressions for the moving frame or the joint invariants.

For invariant finite difference equations that approximate differential equations, the implementation of the group foliation method provides a new type of invariant numerical scheme. This naturally leads to new questions concerning the accuracy and stability of such schemes. As a preliminary investigation, we first consider in Section 7 the Schwarzian differential equation

$$\frac{y_{xxx}}{y_x} - \frac{3}{2}\left(\frac{y_{xx}}{y_x}\right)^2 = F(x),$$

which is invariant under the group of special linear fractional transformations

$$X = x, \qquad Y = \frac{ay+b}{cy+d}, \qquad ad - bc = 1,$$

and prove a discrete analogue of the Schwarz Theorem for an invariant discretization of this equation. Continuing this example, we perform in Section 8 a numerical simulation based on an exact solution of the Schwarz equation that admits vertical asymptotes. In accordance with other numerical simulations using symmetry-preserving schemes, [2,3,7], the group foliation scheme has no difficulty integrating beyond the vertical asymptotes. However, via the group foliation scheme, this unexpected behavior can be clearly explained, at least for the problem at hand. In light of our numerical simulation, we expect that further applications of the group foliation method to symmetry-preserving schemes might shed some light on the numerical properties of those schemes.

## 2 Preliminaries

Let $M$ be an $m$-dimensional manifold with local coordinates $z = (z^1, \ldots, z^m)$. Given a $p$-dimensional submanifold $S \subset M$ with $1 \leq p < m$, let $z = z(s)$ be a local parametrization of $S$ with independent variable $s = (s^1, \ldots, s^p)$. For each integer $0 \leq n \leq \infty$, let $\mathrm{J}^{(n)} = \mathrm{J}^{(n)}(M, p)$ denote the $n^{\text{th}}$ order submanifold jet bundle defined as the set of equivalence classes under the equivalence relation of $n^{\text{th}}$ order contact, [28]. Local coordinates on $\mathrm{J}^{(n)}$ are given by

$$(s, z^{(n)}) = (s, \ \ldots \ z^a_{s^B} \ \ldots) \qquad a = 1, \ldots, m, \quad 0 \leq \#B \leq n,$$

where $z^{(n)}$ indicates the collection of submanifold jet coordinates $z^a_{s^B}$ representing the derivatives $\partial^k z^a / (\partial z^1)^{b^1} \cdots (\partial z^p)^{b^p}$, where $B = (b^1, \ldots, b^p)$ is an ordered multi-index of order $\#B = k$ with nonnegative components $b^\nu \geq 0$.

In the discrete setting, the continuous variable $s = (s^1, \ldots, s^p) \in \mathbb{R}^p$ is replaced by an integer multi-index

$$N = (n^1, \ldots, n^p) \in \mathbb{Z}^p \subset \mathbb{R}^p.$$

For each $N \in \mathbb{Z}^p$, let

$$z_N = z(N)$$

denote a point on the submanifold $S \subset M$. A discrete counterpart to the submanifold jet space $\mathrm{J}^{(n)}$ is given by the $n^{th}$ *order forward discrete jet space* $\mathrm{J}^{[n]}$ with coordinates

$$(N, z^{[n]}_N) = (N, \ \ldots \ z_{N+K} \ \ldots) \qquad N \in \mathbb{Z}^p, \quad 0 \leq \#K \leq n, \tag{2.1}$$

where $z^{[n]}_N$ indicates the collection of points $z_{N+K}$ with $K \in \mathbb{Z}^p_{\geq 0}$ a nonnegative integer multi-index of order at most $n$.

**Remark 2.1.** There are a multitude of ways to approximate the $n^{\text{th}}$ order submanifold jet space $\mathrm{J}^{(n)}$. One can use forward, backward, or centered difference approximations, and in numerical applications one might consider more points for greater accuracy. To simplify the theoretical exposition, we will restrict our attention to forward discrete jets. Adapting the discussion to backward discrete jets or to a mix of forward and backward jets is accomplished by allowing the multi-index $K \in \mathbb{Z}^p$ to contain integer values and by considering discrete jets $(N, z^{[n]}_N) = (N, \ \ldots \ z_{N+K} \ \ldots)$ with at least $d_n$ points $z_{N+K} \in M$ appropriately[1] chosen, where

$$d_n = \binom{p+n}{n}$$

is the number of points $z_{N+K}$ in the discrete jet (2.1). Finally, we note that in numerical analysis the collection of points $z^{[n]}_N$ is also called a lattice.

Let $\pi_n \colon \mathrm{J}^{[n]} \to \mathbb{Z}^p$ be the projection map onto the discrete index

$$\pi_n(N, z^{[n]}_N) = N.$$

We introduce the notation

$$\mathrm{J}^{[n]}|_N = \pi_n^{-1}(N)$$

to denote the fiber over the point $N \in \mathbb{Z}^p$. Note that $\mathrm{J}^{[n]}|_N$ is locally isomorphic to an open subset of $\mathbb{R}^{m \cdot d_n}$, where $m = \dim M$.

[1]In the terminology of [24], the points $z^{[n]}_N$ need to form an $n$-corner lattice.

**Definition 2.2.** A finite difference equation of order $n$ is an equation of the form

$$E(N, z_N^{[n]}) = E(N, \ldots\, z_{N+K}\, \ldots) = 0, \tag{2.2}$$

which depends explicitly on $z_N$ and at least one point $z_{N+K}$ with $\#K = n$.

Given a Lie group $G$ acting smoothly on $M$, the induced action of $g \in G$ on the discrete jet $z_N^{[n]}$ is given by the product action

$$Z_N^{[n]} = g \cdot z_N^{[n]} = (\ldots\, g \cdot z_{N+K}\, \ldots),$$

wherever the action is defined. The induced action on the multi-index $N$ is taken to be trivial:

$$g \cdot N = N.$$

The induced action of the Lie group $G$ on a discrete functions $F\colon \mathrm{J}^{[n]} \to \mathbb{R}$ is given by

$$g \cdot F(N, z_N^{[n]}) = F(N, g \cdot z_N^{[n]}).$$

**Definition 2.3.** A function $I\colon \mathrm{J}^{[n]} \to \mathbb{R}$ is said to be an $n^{\mathrm{th}}$ order *joint invariant* of the group $G$ if

$$g \cdot I(N, z_N^{[n]}) = I(N, g \cdot z_N^{[n]}) = I(N, z_N^{[n]})$$

for all $g \in G$ where the product action is defined.

**Definition 2.4.** A Lie group $G$ is a *symmetry group* of the finite difference equation $E(N, z_N^{(n)}) = 0$ if, wherever the product action is defined,

$$E(N, g \cdot z_N^{[n]}) = 0 \qquad \text{whenever} \qquad E(N, z_N^{[n]}) = 0.$$

We then say that the equation $E(N, z_N^{[n]}) = 0$ is *$G$-invariant.*

Symmetry groups of finite difference equations are found by computing the *(infinitesimal) determining equations* of the symmetry Lie algebra in a fashion similar to the continuous case. We refer to [14, 18, 19] for more details and examples.

**Definition 2.5.** A finite difference equation $E(N, z_N^{(n)}) = 0$ is said to be *strongly $G$-invariant* if $E(N, z_N^{[n]})$ is a joint invariant.

**Remark 2.6.** The condition of being a strongly invariant equation is more restrictive than being an invariant equation. Invariance need only hold on the solution space of the equation, while strong invariance requires invariance on every locus of $E(N, z_N^{[n]})$. Equations that are invariant but not strongly invariant can sometimes be made strongly invariant. If $E(N, z_N^{[n]}) = 0$ is invariant but not strongly invariant, then it must satisfy the equality $E(N, g \cdot z_N^{[n]}) = \mu(g, N, z_N^{[n]}) E(N, g \cdot z_N^{[n]})$ with $\mu \neq 0$. The function $E(N, z_N^{[n]})$ is called a *relative invariant* of weight $\mu$, [29]. If there exists in turn a relative invariant $R(N, z_N^{[n]}) \neq 0$ of weight $1/\mu$, then $R(N, z_N^{[n]}) E(N, z_N^{[n]}) = 0$ is a strongly invariant equation with the same solution space as the original equation.

Finite difference equations frequently occur as approximations of differential equations. As the next example shows, when a differential equation admits a symmetry group it is possible construct approximations that preserve these symmetries.

**Example 2.7.** Let $k$, $a \neq -1$, and $b \neq 1$ be three constants, and consider the ordinary differential equation

$$y' = (k + x^a)y^b. \tag{2.3}$$

This equation admits a 1-dimensional symmetry group whose action on $(x, y)$ is given by

$$X = x + \epsilon, \qquad \frac{Y^{1-b}}{b-1} = \frac{y^{1-b}}{b-1} + \frac{x^{a+1}}{a+1} - \frac{(x+\epsilon)^{a+1}}{a+1}, \qquad \epsilon \in \mathbb{R}.$$

Following standard algorithms presented in [2, 8, 15, 20, 30, 35–37], an invariant discretization of (2.3) is given by

$$\left(\frac{x_{n+1}^{a+1}}{a+1} + \frac{y_{n+1}^{1-b}}{b-1}\right) - \left(\frac{x_n^{a+1}}{a+1} + \frac{y_n^{1-b}}{b-1}\right) + k(x_{n+1} - x_n) = 0, \qquad x_{n+1} - x_n = h, \tag{2.4}$$

where $n \in \mathbb{Z}$ is a one-dimensional subscript and $h > 0$ is a positive constant. The first equation in (2.4) provides an approximation of the differential equation (2.3) while the second equation specifies the mesh. The numerical scheme (2.4) is invariant under the product action

$$X_n = x_n + \epsilon, \qquad \frac{Y_n^{1-b}}{b-1} = \frac{y_n^{1-b}}{b-1} + \frac{x_n^{a+1}}{a+1} - \frac{(x_n+\epsilon)^{a+1}}{a+1}, \qquad \epsilon \in \mathbb{R}, \tag{2.5}$$

where $n \in \mathbb{Z}$.

# 3 Moving frames

Given an $r$-dimensional Lie group $G$ acting on $M$, we consider its product action on the discrete jet space $\mathrm{J}^{[n]}$ and use the method of equivariant moving frames to construct joint invariants.

**Definition 3.1.** An $n^{th}$ *order right joint moving frame* is a $G$-equivariant map[2] $\rho\colon \mathrm{J}^{[n]} \to G$ such that for $g \in G$

$$\rho(N, g \cdot z_N^{[n]}) = \rho(N, z_N^{[n]})\, g^{-1} \tag{3.1}$$

whenever the product action is defined.

Given a right moving frame $\rho\colon \mathrm{J}^{[n]} \to G$, a corresponding *left moving frame* $\overline{\rho}\colon \mathrm{J}^{[n]} \to G$ is obtained by group inversion

$$\overline{\rho}(N, z_N^{[n]}) = \rho(N, z_N^{[n]})^{-1}.$$

To guarantee the existence of a moving frame, mild assumptions on the product group action are imposed.

**Definition 3.2.** A Lie group $G$ is said to act *freely* at a point $z_N^{[n]}$ if the isotropy subgroup

$$G_{z_N^{[n]}} = \{g \in G \,|\, g \cdot z_N^{[n]} = z_N^{[n]}\}$$

consists only of the identity element. The product action is *locally free* at $z_N^{[n]}$ if the isotropy subgroup is discrete. The Lie group $G$ is said to act (locally) freely on $\mathrm{J}^{[n]}$ if for all $N \in \mathbb{Z}^p$ and any point $z_N^{[n]} \in \mathrm{J}^{[n]}|_N$, $G$ acts (locally) freely at $z_N^{[n]}$.

[2] It would be customary to use the notation $\rho^{[n]}$ to denote the $n^{\text{th}}$ order right joint moving frame. To simplify the notation we omit the superscript and let $\rho = \rho^{[n]}$.

**Definition 3.3.** The product action is said to be *regular* at $N \in \mathbb{Z}^p$ if the group orbits in the fiber $\mathrm{J}^{[n]}|_N$ form a regular foliation.

**Theorem 3.4.** A moving frame exists in a neighborhood $\mathcal{V}^{[n]}|_N \subset \mathrm{J}^{[n]}|_N$ of a discrete jet $z_N^{[n]} \in \mathrm{J}^{[n]}|_N$ if and only if $G$ acts locally freely and regularly on $\mathcal{V}^{[n]}|_N$.

If the product action is not free on $\mathrm{J}^{[n]}$, then one should increase the order of the discrete jet space. Under a mild condition on the group action, it was shown in [5] that the product action will eventually become (locally) free on a sufficiently high order discrete jet space.

**Definition 3.5.** A Lie group $G$ acting on $M$ is said to act *effectively on subsets* if, for any open subset $U \subset M$, the global isotropy subgroup of $U$

$$G_U^\star = \{g \in G \mid g \cdot z = z \text{ for all } z \in U\}$$

consists only of the identity element. The Lie group $G$ is said to act *locally effectively on subsets* if, for any open subset $U \subset M$, $G_U^\star$ is a discrete subgroup of $G$.

**Theorem 3.6.** If $G$ acts (locally) effectively on subsets of $M$, then for a fixed multi-index $N \in \mathbb{Z}^p$, there exists $k \geq 0$ such that for all $n \geq k$, the product action of $G$ is locally free on an open dense subset $\mathcal{V}^{[n]}|_N \subset \mathrm{J}^{[n]}|_N$.

In the following we assume that the order $k$ in Theorem 3.6 is the same for all $N \in \mathbb{Z}^p$. Then, the product action is locally free on the discrete jet space $\mathrm{J}^{[n]}$ provided $n \geq k$.

In the discrete setting, a subset $\mathcal{K} \subset \mathrm{J}^{[n]}$ is called a *cross-section* to the group orbits, if for each $N \in \mathbb{Z}^p$, the restriction $\mathcal{K}|_N \subset \mathrm{J}^{[n]}|_N$ is a cross-section in the usual sense. That is, $\mathcal{K}|_N$ is a submanifold of complementary dimension to the group orbits, intersecting the orbits transversally, [10]. In applications, a right moving frame is constructed using the following theorem.

**Theorem 3.7.** If $G$ acts freely and regularly on $\mathrm{J}^{[n]}$ and $\mathcal{K} \subset \mathrm{J}^{[n]}$ is a cross-section to the group orbits, then the map $\rho\colon \mathrm{J}^{[n]} \to G$ whose value at $(N, z_N^{[n]}) \in \mathrm{J}^{[n]}$ is the unique group element $g = \rho(N, z_N^{[n]})$ sending $z_N^{[n]}$ onto the cross-section, i.e. $\rho(N, z_N^{[n]}) \cdot z_N^{[n]} \in \mathcal{K}|_N$, is a right moving frame.

It is convenient, but not necessary, to assume that

$$\mathcal{K} = \{z_{N_1}^{a_1} = c^1, \ldots, z_{N_r}^{a_r} = c^r\}$$

is a coordinate cross-section obtained by setting $r = \dim G$ coordinates of the discrete jet $z_N^{[n]}$ equal to suitable constants. The right moving frame $\rho(N, z_N^{[n]})$ is then obtained by solving the *normalization equations*

$$Z_{N_1}^{a_1} = g \cdot z_{N_1}^{a_1} = c^1, \qquad \ldots \qquad Z_{N_r}^{a_r} = g \cdot z_{N_r}^{a_r} = c^r, \tag{3.2}$$

for the group parameters $g = (g^1, \ldots, g^r)$ in terms of $(N, z_N^{[n]})$. Note that the moving frame may depend explicitly on the multi-index $N$ if the normalization equations (3.2) involve the multi-index $N$, which may happen if the group action depends on $N$. To simplify the notation, we write

$$\rho_N = \rho(N, z_N^{[n]})$$

to denote the value of the moving frame $\rho\colon \mathrm{J}^{[n]} \to G$ at $(N, z_N^{[n]})$.

**Definition 3.8.** Let $\rho\colon \mathrm{J}^{[n]} \to G$ be a right moving frame. The *invariantization* of a discrete function $F\colon \mathrm{J}^{[n]} \to \mathbb{R}$ is the joint invariant

$$\iota_N(F)(N, z_N^{[n]}) = F(N, \rho_N \cdot z_N^{[n]}). \tag{3.3}$$

The proof that (3.3) is a joint invariant follows from the $G$-equivariance property (3.1) of the right moving frame:

$$\begin{aligned} g \cdot \iota_N(F)(N, z_N^{[n]}) &= F(N, \rho(N, g \cdot z_N^{[n]}) \cdot g \cdot z_N^{[n]}) \\ &= F(N, \rho(N, z_N^{[n]}) \cdot g^{-1} \cdot g \cdot z_N^{[n]}) \\ &= \iota_N(F)(N, z_N^{[n]}). \end{aligned}$$

Of particular interest to us will be the invariantization of the coordinate function of a point. The invariantization of the coordinates of $z_L$ with respect to the moving frame $\rho_N$,

$$\iota_N(z_L) = \rho_N \cdot z_L,$$

are called *normalized joint invariants*. By construction, the invariantization with respect to $\rho_N$ of the coordinates defining the normalization equations (3.2) yields constant invariants:

$$\iota_N(z_{N_1}^{a_1}) = c^1 \qquad \ldots \qquad \iota_N(z_{N_r}^{a_r}) = c^r. \tag{3.4}$$

These constant invariants are called *phantom invariants*.

An important fact that will be useful in the group foliation method is that the invariantization of a joint invariant $I(N, z_N^{[n]})$ equals the invariant itself:

$$\iota_N(I) = I. \tag{3.5}$$

This is known as the *replacement principle*, [23], since in (3.5) the discrete jet $z_N^{[n]}$ is replaced by the normalized invariants $\iota_N(z_N^{[n]})$ producing the equality

$$I(N, z_N^{[n]}) = I(N, \iota_N(z_N^{[n]})).$$

**Example 3.9.** As a simple illustration of the moving frame construction, we consider the product action (2.5). Since the action is free and regular on $\mathrm{J}^{[0]} = \{(n, x_n, y_n)\}$, a joint moving frame can be constructed by choosing the cross-section $\mathcal{K} = \{x_n = 0\}$. Solving the normalization equation $0 = X_n = x_n + \epsilon$ for the group parameter $\epsilon$ we obtain the right moving frame

$$\rho_n\colon \qquad \epsilon_n = -x_n. \tag{3.6}$$

We add the subscript $n$ to the group parameter to emphasize its dependence on the discrete index. With a moving frame in hand, we can invariantize the coordinate functions $(x_k, y_k)$, $k \in \mathbb{Z}$. For example,

$$
\begin{aligned}
0 = \iota_n(x_n) &= X_n\Big|_{\epsilon_n=-x_n} = x_n - x_n, \\
H_n = \iota_n(x_{n+1}) &= X_{n+1}\Big|_{\epsilon_n=-x_n} = x_{n+1} - x_n, \\
I_n = \iota_n(x_{n+2}) &= X_{n+2}\Big|_{\epsilon_n=-x_n} = x_{n+2} - x_n, \\
J_n = \iota_n\left(\frac{y_n^{1-b}}{b-1}\right) &= \frac{Y_n^{1-b}}{b-1}\Big|_{\epsilon_n=-x_n} = \frac{y_n^{1-b}}{b-1} + \frac{x_n^{a+1}}{a+1}, \\
K_n = \iota_n\left(\frac{y_{n+1}^{1-b}}{b-1}\right) &= \frac{Y_{n+1}^{1-b}}{b-1}\Big|_{\epsilon_n=-x_n} = \frac{y_{n+1}^{1-b}}{b-1} + \frac{x_{n+1}^{a+1}}{a+1} - \frac{(x_{n+1}-x_n)^{a+1}}{a+1}, \\
L_n = \iota_n\left(\frac{y_{n+2}^{1-b}}{b-1}\right) &= \frac{Y_{n+2}^{1-b}}{b-1}\Big|_{\epsilon_n=-x_n} = \frac{y_{n+2}^{1-b}}{b-1} + \frac{x_{n+2}^{a+1}}{a+1} - \frac{(x_{n+2}-x_n)^{a+1}}{a+1}.
\end{aligned}
\tag{3.7}
$$

When $m \neq n$, note that the invariantizations of the coordinate functions with respect to moving frames $\rho_m$ and $\rho_n$ will differ. For example, $\iota_n(x_n) = 0$, but $\iota_m(x_n) = x_n - x_m \neq 0$. Similarly,

$$\iota_m\left(\frac{y_n^{1-b}}{b-1}\right) - \iota_n\left(\frac{y_n^{1-b}}{b-1}\right) = -\frac{(x_n - x_m)^{a+1}}{a+1} \neq 0.$$

Providing a relationship between these different invariantizations is the purpose of the *discrete recurrence relations* introduced in the next section.

# 4 Recurrence relations

In the discrete setting, the natural operators on discrete functions are provided by the forward and backward shift operators. Introducing the notation

$$1_i = (0, \ldots, 0, 1, 0, \ldots, 0) \in \mathbb{R}^p$$

for the $i^{\text{th}}$ standard basis element with a 1 in the $i^{\text{th}}$ component and 0 elsewhere, the forward and backward shift operators $\mathbf{S}_i^{\pm} \colon \mathbb{Z}^p \to \mathbb{Z}^p$ on the multi-index $N = (n^1, \ldots, n^p)$ are given by setting

$$\mathbf{S}_i^{\pm}(N) = N \pm 1_i = (n^1, \ldots, n^i \pm 1, \ldots, n^p), \qquad i = 1, \ldots, p.$$

The action of the shift operators $\mathbf{S}_i^{\pm}$ extends to discrete functions by letting

$$\mathbf{S}_i^{\pm}[F(N, z_N^{[n]})] = F(N \pm 1_i, z_{N+1_i}^{[n]}).$$

Given a multi-index $K = (k^1, \ldots, k^p) \in \mathbb{Z}^p$, we introduce the notation

$$\mathbf{S}^K = \mathbf{S}_1^{k^1} \circ \cdots \circ \mathbf{S}_p^{k^p}$$

for the composition of shift operators, with the convention that $\mathbf{S}_i^0 = \mathbb{1}$ is the identity map and

$$\mathbf{S}_i^{k^i} = \underbrace{\mathbf{S}_i^+ \circ \cdots \circ \mathbf{S}_i^+}_{|k_i| \text{ times}} \quad \text{if} \quad k^i > 0 \qquad \text{and} \qquad \mathbf{S}_i^{k^i} = \underbrace{\mathbf{S}_i^- \circ \cdots \circ \mathbf{S}_i^-}_{|k_i| \text{ times}} \quad \text{if} \quad k^i < 0.$$

The invariantization of a discrete function $F\colon \mathrm{J}^{[n]} \to \mathbb{R}$ with respect to the moving frame $\rho_N$ or its shift $\mathbf{S}^K(\rho_N) = \rho_{N+K}$ will, in general, produce different joint invariants:

$$\iota_N(F) \neq \iota_{N+K}(F).$$

The purpose of the recurrence relations is to provide the relationship between these normalized invariants. This relationship is analogous to that of the normalized differential invariants and their invariant derivatives provided by the recurrence relations in the continuous theory, [10].

**Definition 4.1.** Let $\rho\colon \mathrm{J}^{[n]} \to G$ be a right moving frame. The $i^{th}$ *Maurer–Cartan invariant* is the group element

$$\mathfrak{m}_N^i = \rho_N\, \rho_{N+1_i}^{-1} \in G, \qquad i = 1, \ldots, p. \tag{4.1}$$

The invariance of $\mathfrak{m}_N^i$ follows from the right-equivariance of $\rho_N$ and $\rho_{N+1_i}$:

$$\begin{aligned} \mathfrak{m}_N^i(N, g\cdot z_N^{[n+1]}) &= \rho(N, g\cdot z_N^{[n]})\, \rho^{-1}(N+1_i, g\cdot z_{N+1_i}^{[n]}) \\ &= \rho(N, z_N^{[n]})\, g^{-1}\, g\, \rho^{-1}(N+1_i, z_{N+1_i}^{[n]}) \\ &= \mathfrak{m}_N^i(N, z_N^{[n+1]}). \end{aligned}$$

**Proposition 4.2.** Let $F\colon \mathrm{J}^{[n]} \to \mathbb{R}$ be a discrete function. The joint invariants $\iota_N(F)$ and $\iota_{N+1_i}(F)$ satisfy the relation

$$\iota_N(F) = \mathfrak{m}_N^i \cdot \iota_{N+1_i}(F), \qquad i = 1, \ldots, p, \tag{4.2a}$$

known as the *recurrence relation* between these normalized invariants. Similarly, the recurrence relation for the joint invariants $\iota_N(F)$ and $\iota_{N-1_i}(F)$ is

$$\iota_N(F) = (\mathfrak{m}_{N-1_i}^i)^{-1} \cdot \iota_{N-1_i}(F), \qquad i = 1, \ldots, p. \tag{4.2b}$$

**Remark 4.3.** To compute the right-hand side of (4.2a), which is given by

$$\mathfrak{m}_N^i \cdot \iota_{N+1_i}(F) = F(N, \mathfrak{m}_N^i \cdot \iota_{N+1_i}(z_N^{[n]})),$$

the invariant $\iota_{N+1_i}(z_N^{[n]})$ should be identified with its corresponding point on the cross-section $\mathcal{K}$. After this identification, we let the group element $\mathfrak{m}_N^i$ act on the point $\iota_{N+1_i}(z_N^{[n]})$. The computation of (4.2b) is performed in a similar fashion.

*Proof.* Equation (4.2a) follows from a direct computation:

$$\begin{aligned} \iota_N(F) &= F(N, \rho_N \cdot z_N^{[n]}) \\ &= F(N, (\rho_N\, \rho_{N+1_i}^{-1}) \cdot \rho_{N+1_i} \cdot z_N^{[n]}) \\ &= \mathfrak{m}_N^i \cdot \iota_{N+1_i}(F). \end{aligned}$$

Equation (4.2b) follows from a similar computation. Alternatively, from (4.2a), with the function $F$ replaced by its shift $\mathbf{S}_i^+(F)$, one has

$$\iota_{N+1_i}(\mathbf{S}_i^+(F)) = (\mathfrak{m}_N^i)^{-1} \cdot \iota_N(\mathbf{S}_i^+(F)).$$

Applying the negative shift operator $\mathbf{S}_i^-$ to the formula yields (4.2b). □

In the following we will be primarily interested in the case when the discrete function is the coordinate function of a point. Substituting $F = z_{L+1_i}$ into the recurrence relation (4.2a) we obtain

$$\iota_N(z_{L+1_i}) = \mathfrak{m}_N^i \cdot \iota_{N+1_i}(z_{L+1_i}) = \mathfrak{m}_N^i \cdot \mathbf{S}_i^+[\iota_N(z_L)], \qquad i = 1, \ldots, p. \tag{4.3a}$$

The latter provides an expression for the invariantization of the $i^{\text{th}}$ forward shift of $z_L$ in terms of the $i^{\text{th}}$ Maurer–Cartan invariant and the $i^{\text{th}}$ forward shift of the normalized invariant $\iota_N(z_L)$. Similarly, substituting $F = z_{L-1_i}$ in (4.2b) yields

$$\iota_N(z_{L-1_i}) = \mathfrak{m}_{N-1_i}^i \cdot \iota_{N-1_i}(z_{L-1_i}) = \mathfrak{m}_{N-1_i}^i \cdot \mathbf{S}_i^-[\iota_N(z_L)], \qquad i = 1, \ldots, p. \tag{4.3b}$$

**Example 4.4.** We now compute the recurrence relations (4.3) for the product action (2.5) of Example 2.7. Recall that the expressions for the moving frame and some of its normalized joint invariants were obtained in Example 3.9.

From Definition 4.1 and the moving frame expression (3.6), the Maurer–Cartan invariant is

$$\mathfrak{m}_n = \rho_n\, \rho_{n+1}^{-1} = \epsilon_n - \epsilon_{n+1} = x_{n+1} - x_n = \iota_n(x_{n+1}) = H_n. \tag{4.4}$$

Applying the recurrence relation (4.3a) to the normalized invariant $I_n = \iota_n(x_{n+2})$ computed in (3.7), we obtain

$$I_n = \iota_n(x_{n+2}) = \mathfrak{m}_n \cdot \iota_{n+1}(x_{n+2}) = \mathfrak{m}_n \cdot H_{n+1} = H_n \cdot H_{n+1} = H_n + H_{n+1}, \tag{4.5}$$

where

$$H_{n+1} = \iota_{n+1}(x_{n+2}) = x_{n+2} - x_{n+1} = \mathbf{S}^+(H_n)$$

is the joint invariant obtained by shifting $H_n$ forward once. To compute the product $H_n \cdot H_{n+1}$ in (4.5) we used the transformation rule for $x_n$ in (2.5) (replacing $x_n$ by $H_{n+1}$ and $\epsilon$ by $H_n$) since $H_{n+1} = \iota_{n+1}(x_{n+2})$ is obtained by invariantizing the independent variable $x_{n+2}$. Similarly,

$$\begin{aligned} K_n &= \iota_n\left(\frac{y_{n+1}^{1-b}}{b-1}\right) = \mathfrak{m}_n \cdot \iota_{n+1}\left(\frac{y_{n+1}^{1-b}}{b-1}\right) = \mathfrak{m}_n \cdot J_{n+1} = H_n \cdot J_{n+1} = J_{n+1} - \frac{H_n^{a+1}}{a+1}, \\ L_n &= \iota_n\left(\frac{y_{n+2}^{1-b}}{b-1}\right) = \mathfrak{m}_{n+1} \cdot \mathfrak{m}_n \cdot \iota_{n+2}\left(\frac{y_{n+2}^{1-b}}{b-1}\right) = (\mathfrak{m}_{n+1}\,\mathfrak{m}_n) \cdot J_{n+2} = (H_{n+1} \cdot H_n) \cdot J_{n+2} \\ &= (H_n + H_{n+1}) \cdot J_{n+2} = J_{n+2} - \frac{(H_n + H_{n+1})^{a+1}}{a+1}. \end{aligned}$$

In general, for $k \geq 1$, we have

$$\iota_n(x_{n+k}) = \sum_{\ell=1}^{k} H_{n+\ell-1}, \qquad \iota_n \cdot \left(\frac{y_{n+k}^{1-b}}{b-1}\right) = J_{n+k} - \frac{1}{a+1}\left(\sum_{\ell=1}^{k} H_{n+\ell-1}\right)^{a+1}, \tag{4.6a}$$

and for $k \leq -1$

$$\iota_n(x_{n+k}) = -\sum_{\ell=1}^{-k} H_{n-\ell}, \qquad \iota_n(y_{n+k}) = J_{n+k} - \frac{1}{a+1}\left(\sum_{\ell=1}^{-k} H_{n-\ell}\right)^{a+1}. \tag{4.6b}$$

From the recurrence relations (4.6), we observe that all the normalized joint invariants $\iota_n(x_{n+k})$, $\iota_n(y_{n+k})$, $k \in \mathbb{Z}$, can be expressed in terms of the invariants $H_n$, $J_n$, and their shifts. As we explain in the next section, the invariants $H_n$, $J_n$ are said to generate the algebra of joint invariants.

As in the continuous theory of equivariant moving frames, [10], the coordinate expressions for the moving frame $\rho_N$ are not required to compute the Maurer–Cartan invariants $\mathfrak{m}_N^i$ symbolically. The only data needed is the choice of a cross-section and the expression for the group action. To obtain $\mathfrak{m}_N^i$, shift the phantom invariants (3.4) by $\mathbf{S}_i^+$ to obtain

$$\rho_{N+1_i} \cdot z^{a_1}_{N_1+1_i} = c^1, \qquad \ldots \qquad \rho_{N+1_i} \cdot z^{a_r}_{N_r+1_i} = c^r.$$

Inserting the identity element $\rho_N^{-1}\,\rho_N$ on the left-hand side of each equality yields

$$(\mathfrak{m}_N^i)^{-1} \cdot \iota_N(z^{a_1}_{N_1+1_i}) = c^1, \qquad \ldots \qquad (\mathfrak{m}_N^i)^{-1} \cdot \iota_N(z^{a_r}_{N_r+1_i}) = c^r. \tag{4.7}$$

By assumption, since the normalization equations (3.2) can be solved for the group parameters to obtain the right moving frame $\rho_N$, the equations (4.7) can be solved for $(\mathfrak{m}_N^i)^{-1}$. Inverting the group element we obtain symbolic expressions for $\mathfrak{m}_N^i$ in terms of the normalized invariants $\iota_N(z^{a_1}_{N_1+1_i}), \ldots, \iota_N(z^{a_r}_{N_r+1_i})$, and possibly $N$.

**Example 4.5.** Recalling Example 4.4, the Maurer–Cartan invariant (4.4) can be deduced symbolically as follows. Taking the forward shift of the phantom invariant $\rho_n \cdot x_n = 0$, we have

$$0 = \rho_{n+1} \cdot x_{n+1} = (\rho_{n+1}\,\rho_n^{-1}) \cdot \rho_n \cdot x_{n+1} = \mathfrak{m}_n^{-1} \cdot H_n = H_n - \mathfrak{m}_n.$$

Solving for the Maurer–Cartan invariant $\mathfrak{m}_n$, we obtain

$$\mathfrak{m}_n = H_n.$$

As it was done in Example 4.4, the recurrence relations (4.2) or (4.3) can be iterated. For the ordered multi-index $K = (k^1, \ldots, k^p) \in \mathbb{Z}^p$ of order $\#K = n = |k^1| + \cdots + |k^p|$, consider an unordered (symmetric) multi-index

$$\widetilde{K} = (\widetilde{k}^1, \ldots, \widetilde{k}^n) \in \mathbb{Z}^n \qquad \text{with} \qquad 1 \leq |\widetilde{k}^\ell| \leq p,$$

such that $|k^1|$ components are equal to $\operatorname{sign}(k^1) \cdot 1$, $|k^2|$ components are equal to $\operatorname{sign}(k^2) \cdot 2$, $\ldots$, and $|k^p|$ components are equal to $\operatorname{sign}(k^p) \cdot p$. For example, given the 2-dimensional multi-index $K = (-2, 3)$, we can consider the unordered multi-index

$$\widetilde{K} = (-1, 2, 2, 2, -1).$$

Given a multi-index $K$ and a chosen unordered multi-index $\widetilde{K}$, the iteration of the recurrence relations (4.3) according to the unordered multi-index $\widetilde{K}$ yields the recurrence formula

$$\iota_N(z_{N+K}) = \widetilde{\mathfrak{m}}_N^{\widetilde{k}^1}\, \widetilde{\mathfrak{m}}^{\widetilde{k}^2}_{N,\widetilde{k}^1} \cdots \widetilde{\mathfrak{m}}^{\widetilde{k}^n}_{N,\widetilde{k}^1,\ldots,\widetilde{k}^{n-1}} \cdot \iota_{N+K}(z_{N+K}), \tag{4.8}$$

where

$$\widetilde{\mathfrak{m}}^{\widetilde{k}^\ell}_{N,\widetilde{k}^1,\ldots,\widetilde{k}^{\ell-1}} = \begin{cases} \mathfrak{m}^{\widetilde{k}^\ell}_{N+\sum_{i=1}^{\ell-1}\operatorname{sign}(\widetilde{k}^i)\cdot 1_{|\widetilde{k}^i|}} & \text{if} \quad \widetilde{k}^\ell > 0, \\ \left(\mathfrak{m}^{|\widetilde{k}^\ell|}_{N+(\sum_{i=1}^{\ell-1}\operatorname{sign}(\widetilde{k}^i)\cdot 1_{|\widetilde{k}^i|})-1_{|\widetilde{k}^\ell|}}\right)^{-1} & \text{if} \quad \widetilde{k}^\ell < 0. \end{cases}$$

Since group multiplication is not necessarily commutative, replacing the right-hand side of (4.8) by a permutation $\sigma(\widetilde{K})$ will, in general, produce a different expression. But, since the left-hand side would the same in both cases, we obtain the equality

$$\begin{aligned} &\widetilde{\mathfrak{m}}_N^{\widetilde{k}^1}\, \widetilde{\mathfrak{m}}^{\widetilde{k}^2}_{N,\widetilde{k}^1} \cdots \widetilde{\mathfrak{m}}^{\widetilde{k}^n}_{N,\widetilde{k}^1,\ldots,\widetilde{k}^{n-1}} \cdot \iota_{N+K}(z_{N+K}) \\ &\qquad = \widetilde{\mathfrak{m}}_N^{\sigma(\widetilde{k}^1)}\, \widetilde{\mathfrak{m}}^{\sigma(\widetilde{k}^2)}_{N,\sigma(\widetilde{k}^1)} \cdots \widetilde{\mathfrak{m}}^{\sigma(\widetilde{k}^n)}_{N,\sigma(\widetilde{k}^1),\ldots,\sigma(\widetilde{k}^{n-1})} \cdot \iota_{N+K}(z_{N+K}). \end{aligned}$$

This last equation is called a syzygy (see Definition 5.8).

# 5 Algebra of joint invariants

Essential to the implementation of the group foliation method is the existence of a set of joint invariants that generates the algebra of invariants.

**Definition 5.1.** A set of joint invariants $\mathbf{I}_{\text{gen}} = \{N, \ldots\, I^i_N\, \ldots\}$ is said to *generate* the algebra of joint invariants if any joint invariant $I$ can be expressed as a function of the invariants in $\mathbf{I}_{\text{gen}}$ and their shifts:

$$I = F(N, \ldots\, I^i_{N+K}\, \ldots).$$

In the differential setting, the Basis Theorem (also known as the Lie–Tresse Theorem) guarantees the existence of a finite-dimensional generating set for the algebra of differential invariants; any differential invariant can be expressed in terms of these generating invariants and their invariant derivatives. This theorem was originally proved by Lie for finite-dimensional Lie group actions, [22, p. 760], and extended to infinite-dimensional Lie pseudo-groups by Tresse, [39]. Modern proofs may be found in the classical textbooks [29, 33]. Other proofs based on Spencer cohomology, [17], Weyl algebras, [26], homological methods, [16] or moving frames, [12, 32], also exist.

An analogous Basis Theorem holds for joint invariants, where the invariant derivative operators are replaced by the shift operators $\mathbf{S}^{\pm}_i$.

**Definition 5.2.** A set of invariants $\mathbf{I}$ is said to be complete if any invariant function can be expressed as a function of the invariants contained in $\mathbf{I}$ and their shifts.

**Lemma 5.3.** Let $\rho\colon \mathrm{J}^{[n]} \to G$ be a right moving frame. For any $k \geq 0$, a complete set of joint invariants of order $\leq n{+}k$ is given by the multi-index $N$ and the normalized invariants $\iota_N(z^{[n+k]}_N)$.

*Proof.* Any joint invariant $I$ of order $\leq n + k$ can be expressed in terms of $(N, z^{[n+k]}_N)$. By the replacement principle,

$$I(N, z^{[n+k]}_N) = I(N, \iota_N(z^{[n+k]}_N)).$$

□

**Theorem 5.4.** Suppose $G$ acts freely and regularly on $\mathrm{J}^{[n]}$. Then there exists a finite-dimensional set of invariants $\mathbf{I}_{\text{gen}} = \{N, I^1_N, \ldots, I^s_N\}$, where $I^\ell_N\colon \mathrm{J}^{[n+1]} \to \mathbb{R}$, $\ell = 1, \ldots, s$, are joint invariants of order $\leq n{+}1$, that generates the algebra of invariants.

*Proof.* By assumption, a moving frame exists on $\mathrm{J}^{[n]}$, and by Lemma 5.3 the invariantization of the order $n{+}1$ discrete jet

$$\iota_N(z^{[n+1]}_N) = \rho_N(N, z^{[n]}_N) \cdot z^{[n+1]}_N \tag{5.1}$$

provides a complete set joint invariants of order $\leq n{+}1$. Removing the phantom invariants from (5.1), we show that the remaining normalized invariants $\mathbf{I}_{\text{gen}} = \{N, I^1_N, \ldots, I^s_N\}$ provide a generating set for the algebra of joint invariants. Let

$$\mathbf{I}^{n+1}_{\text{gen}} = \{\iota_N(z_{N+K}) \,|\, \#K = n + 1\} \subset \mathbf{I}_{\text{gen}}$$

denote the subset of normalized invariants obtained by invariantizing the coordinate functions parametrizing the fibers of the projection map $\pi_n^{n+1}\colon \mathrm{J}^{[n+1]} \to \mathrm{J}^{[n]}$. Since the restriction of a moving frame $\rho\colon \mathrm{J}^{[n]} \to G$ to the cross-section $\mathcal{K}$ yields the identity group element, for any invariant $\iota_N(z_{N+K}) \in \mathbf{I}_{\text{gen}}^{n+1}$ we have that

$$\begin{aligned}\mathbf{S}_i^{+}[\iota_N(z_{N+K})]|_{\mathcal{K}} &= [\rho_{N+1_i} \cdot z_{N+K+1_i}]|_{\mathcal{K}} \\ &= e \cdot z_{N+K+1_i}|_{\mathcal{K}} \\ &= z_{N+K+1_i}, \qquad i = 1, \ldots, p.\end{aligned}$$

These equalities imply that the set

$$\bigcup_{i=1}^{p} \mathbf{S}_i^{+}(\mathbf{I}_{\text{gen}}^{n+1})$$

contains $m\binom{p+n+1}{n+2}$ functionally independent invariants, none of which are in $\mathbf{I}_{\text{gen}}$. As a result, the sets

$$\mathbf{I}_{\text{gen}} \bigcup_{i=1}^{p} \mathbf{S}_i^{+}(\mathbf{I}_{\text{gen}}^{n+1}) \qquad \text{and} \qquad \{N, \iota_N(z_N^{[n+2]})\}$$

have the same number of functionally independent invariants. Since the second set provides a complete set of joint invariants of order $\leq n{+}2$, the first set also provides a complete set of invariants of order $\leq n{+}2$. Repeating the argument, the set

$$\mathbf{I}_{\text{gen}} \bigcup_{\#K=1}^{\ell} \mathbf{S}^{K}(\mathbf{I}_{\text{gen}}^{n+1}) \quad \text{where} \quad K \in \mathbb{Z}_{+}^{p},$$

provides a complete set of joint invariants of order $\leq n + \ell + 1$. We therefore conclude that any joint invariant can be expressed in terms of $\mathbf{I}_{\text{gen}}$ and its shifts. □

**Remark 5.5.** In the proof of Theorem 5.4, the generating set $\mathbf{I}_{\text{gen}}$ is not necessarily minimal. It is possible that relations among the generating invariants and their shifts exist, making some of the generating invariants superfluous. Unfortunately, as in the differential setting, we do not have a general algorithm for extracting a minimal generating set. In practice, a minimal generating set may be found by inspection.

**Example 5.6.** In Example 4.4 we found that the normalized joint invariants $\iota_n(x_{n+k})$, $\iota_n(y_{n+k}^{1-b}/(b-1))$ could be expressed in terms of the invariants $H_n$, $J_n$, and their shifts. Therefore, these two invariants, together with the discrete index $n$, form a generating set for the algebra of joint invariants. This generating set is minimal.

In [11], it was shown that in the differential case the algebra of differential invariants is generated by the Maurer–Cartan invariants and the order zero normalized invariants. An analogous result holds for joint invariants, [24].

**Theorem 5.7.** Let $\rho_N$ be a moving frame. The algebra of joint invariants is generated by the multi-index $N$, the Maurer–Cartan invariants $\mathfrak{m}_N^1, \ldots, \mathfrak{m}_N^p$, and the joint invariants $\iota_N(z_N)$.

*Proof.* The proof is a straightforward application of the recurrence relation (4.8). By the replacement principle, [23], any joint invariant may be expressed in terms of the multi-index $N$ and the normalized invariants $\iota_N(z_{N+K})$. By the recurrence formula (4.8), the latter can be re-expressed in terms of the joint invariants $\iota_N(z_N)$, the Maurer–Cartan invariants $\mathfrak{m}_N^1, \ldots, \mathfrak{m}_N^p$, and their shifts. □

We end this section by introducing the notion of syzygy.

**Definition 5.8.** A *syzygy* among the generating invariants $\mathbf{I}_{\text{gen}} = \{N, I_N^1, \ldots, I_N^s\}$ is an identity

$$\mathcal{S}(N, I_N^1, \ldots, I_N^s, \ldots, I_{N+K}^1, \ldots, I_{N+K}^s) \equiv 0$$

relating the invariants $\mathbf{I}_{\text{gen}}$ and their shifts.

We note that shifting a syzygy $\mathcal{S}$ produces a new syzygy. This new syzygy can be considered redundant since it is a direct consequence of $\mathcal{S}$. This leads us to introduce the notion of a generating set of syzygies.

**Definition 5.9.** A *generating set of syzygies* is a collection of syzygies from which all other syzygies can be derived. That is, any syzygy is a function of the generating syzygies and their shifts.

In the differential case, it was shown in [12, 32] that the set of generating syzygies is finite-dimensional. To the best of our knowledge, a similar result in the discrete setting has yet to be proven in full generality. From [24], we can deduce that when the normalized invariants $\iota_N(z_N)$ in Theorem 5.7 are constant and $N \in \mathbb{Z}^p$, then a generating set of syzygies among the Maurer–Cartan invariants is provided by the relations

$$\mathfrak{m}_N^j \, \mathfrak{m}_{N+1_j}^i = \mathfrak{m}_N^i \, \mathfrak{m}_{N+1_i}^j, \qquad 1 \leq i < j \leq p.$$

Examples of syzygies will appear in Example 6.7.

# 6 Group foliation

We now have everything in place to describe the group foliation method for finite difference equations using the method of moving frames. The algorithm and techniques used are similar to the continuous theory developed in [23, 38]. Our starting point is a strongly $G$-invariant finite difference equation

$$E(N, z_N^{[n]}) = 0. \tag{6.1}$$

The method of group foliation uses the foliation of the solution space of (6.1) by the orbits of the group action to decompose $E = 0$ into two alternative systems of finite difference equations, called the *resolving* and *reconstruction* equations. Geometrically, the resolving system is a collection of finite difference equations that the solutions to $E = 0$ must satisfy when projected onto the space of joint invariants. Given a right moving frame $\rho_N$, the resolving system in obtained by first invariantizing the equation $E = 0$ and expressing the resulting equation in terms of a generating set of joint invariants. The resolving system is then completed by appending a generating set of syzygies among the generating invariants. The latter are essential as they provide integrability conditions among the generating invariants. As Ovsiannikov writes in [33], since the resolving system removes the "excess" solutions arising from the symmetry group, the latter should be easier to solve than the original equation (6.1). On the other hand, the reconstruction equations are a collection of first order $G$-automorphic finite difference equations describing the evolution of the left moving frame $\overline{\rho}_N$ along a solution of the resolving system. Once these equations are solved, solutions to the original finite difference equation (6.1) are obtained by acting on solutions of the resolving system with solutions of the reconstruction equations.

The group foliation method has a simple geometric interpretation, illustrated in Figure 1. Intuitively, the invariantization of $E = 0$ (together with the integrability conditions among the invariants) is equivalent to projecting its solutions onto solutions of the resolving system via the action of the right moving frame $\rho_N$. Inversely, the left moving frame $\overline{\rho}_N$, which solves the reconstruction equations, maps solutions of the resolving system back to solutions of $E = 0$.

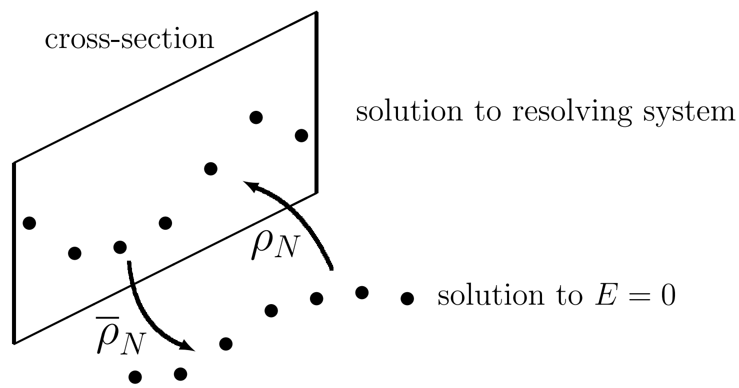



Figure 1: The geometry of group foliation.

We now formally introduce the resolving and reconstruction equations. First, let $\rho_N$ be a right moving frame. Since the equation $E(N, z_N^{[n]}) = 0$ is assumed to be strongly $G$-invariant, the function $E(N, z_N^{[n]})$ is a joint invariant that may be re-expressed in terms of the normalized joint invariants (and the phantom invariants) using the replacement principle, [23],

$$E(N, z_N^{[n]}) = \iota_N[E(N, z_N^{[n]})] = E(N, \iota_N(N, z_N^{[n]})).$$

Let $\mathbf{I}_{\text{gen}} = \{N, I_N^1, \ldots, I_N^s\}$ be a finite generating set of the algebra of joint invariants, ensured by the Basis Theorem 5.4 or Theorem 5.7. The equation $E = 0$ can then be re-rewritten in terms of $\mathbf{I}_{\text{gen}}$ and their shifts:

$$0 = E(N, z_N^{[n]}) = E(N, \iota_N(z_N^{[n]})) = \widetilde{E}(N, \ldots I_{N+K}^1, \ldots, I_{N+K}^s \ldots). \tag{6.2a}$$

The finite difference equation (6.2a) is called the *reduced* or *invariantized equation*. Next, we consider a generating set of syzygies among the generating invariants $\mathbf{I}_{\text{gen}}$:

$$\mathcal{S}(N, \ldots, I_{N+K}^1, \ldots, I_{N+K}^s, \ldots) = 0. \tag{6.2b}$$

The equations (6.2b) are adjoined to (6.2a) as they provide integrability conditions among the generating invariants. Following Vessiot's terminology, the enlarged system of equations (6.2) is called the *resolving system* of $E = 0$ with respect to the symmetry group $G$. Geometrically, solutions to the resolving system are projections onto the space of joint invariants of solutions to the original finite difference equation. This may also be referred to as the (invariant) *signature* of the solution, [6, 31].

**Remark 6.1.** The coordinate expressions for the moving frame and the generating invariants $\mathbf{I}_{\text{gen}}$ are not required to write down the resolving system (6.2). Once a cross-section to the product group action is chosen, the resolving system can be derived symbolically using the replacement principle (3.5) and the recurrence relations (4.3).

Let

$$I_N^i = f^i(N), \qquad i = 1, \ldots, s, \tag{6.3}$$

be a solution of the resolving system (6.2). The next step in the group foliation algorithm is to use the solution (6.3) to recover a solution $z_N$ to the original finite difference equation (6.1). To this end, assume $z_N$ is a solution of (6.1). Then by definition of the left and right moving frames

$$z_N = \rho_N^{-1} \cdot \rho_N \cdot z_N = \overline{\rho}_N \cdot \iota_N(z_N). \tag{6.4}$$

Expressing the order 0 normalized invariant $\iota_N(z_N)$ in terms of the generating invariants $\mathbf{I}_{\text{gen}}$ and substituting the solution (6.3) into the resulting formula, we obtain an explicit expression for $\iota_N(z_N) = H(N)$ in terms of the multi-index $N$. According to (6.4), a solution to the original equation (6.1) is then obtained by acting on $\iota_N(z_N) = H(N)$ by the left moving frame $\overline{\rho}_N$. To determine $\overline{\rho}_N$, we introduce the *reconstruction equations*

$$\begin{aligned} \overline{\rho}_{N+1_i} &= \rho_{N+1_i}^{-1} \\ &= \rho_N^{-1}\, \rho_N\, \rho_{N+1_i}^{-1} \\ &= \overline{\rho}_N\, \mathfrak{m}_N^i, \qquad i = 1, \ldots, p, \end{aligned} \tag{6.5}$$

which prescribe the evolution of the left moving frame $\overline{\rho}_N$ along the solution (6.3). Prior to solving (6.5), the Maurer–Cartan invariants $\mathfrak{m}_N^i$ need to be expressed in terms of the generating invariants $\mathbf{I}_{\text{gen}}$ and their shifts, after which the substitutions (6.3) are made.

**Remark 6.2.** If $G$ is a solvable group with faithful matrix representation in the set of upper triangular matrices, then the Maurer–Cartan invariants (4.1) will also be upper triangular and the reconstruction equations (6.5) can be solved iteratively, starting from the last row and moving up.

We conclude this section by showing that the reconstruction equations (6.5) are $G$-automorphic.

**Definition 6.3.** A system of finite difference equations is called *$G$-automorphic* if all its solutions can be obtained from a single solution via transformations belonging to $G$.

**Proposition 6.4.** The reconstruction equations (6.5) are $G$-automorphic.

*Proof.* Let $\overline{\rho}_N^1$ and $\overline{\rho}_N^2$ be two left moving frames satisfying (6.5). Consider the $G$-valued function

$$g_N = \overline{\rho}_N^1 (\overline{\rho}_N^2)^{-1}. \tag{6.6}$$

Applying the forward shift $\mathbf{S}_i^+$ to $g_N$, and using the reconstruction equations (6.5), we find that

$$\begin{aligned} g_{N+1_i} &= \overline{\rho}^1_{N+1_i}(\overline{\rho}^2_{N+1_i})^{-1} \\ &= \overline{\rho}^1_N\, \mathfrak{m}^i_N(\mathfrak{m}^i_N)^{-1}(\overline{\rho}^2_N)^{-1} \\ &= \overline{\rho}^1_N(\overline{\rho}^2_N)^{-1} \\ &= g_N, \end{aligned}$$

for $i = 1, \ldots, p$. Thus, the function $g_N$ is equal a fixed group element: $g_N = g \in G$. From (6.6), we conclude that

$$\overline{\rho}^1_N = g\, \overline{\rho}^2_N,$$

which shows that (6.5) is $G$-automorphic. □

## 6.1 Examples

We now illustrate the group foliation method with three examples. The first two examples deal with first and second order ordinary finite difference equations while the third involves a partial difference equation.

**Example 6.5.** Finishing Examples 2.7, 3.9, and 4.4, we now apply the group foliation method to the invariant finite difference equations (2.4). As mentioned in Example 5.6, the invariants $\{n, H_n, J_n\}$ form a generating set of the algebra of joint invariants. Expressing the finite difference equations (2.4) in terms of these generating invariants and their shifts we obtain the reduced equations

$$J_{n+1} - J_n + k\, H_n = 0, \qquad H_n = h. \tag{6.7}$$

Since there is no syzygy among the invariants $H_n$, $J_n$, and their shifts, the resolving system is therefore given by (6.7). These equations are easily solved. The second equation implies that the invariant

$$H_n = h \tag{6.8a}$$

is constant. The first equation is a linear first order difference equation whose solution is

$$J_n = J_0 - (k\, h)n, \tag{6.8b}$$

where $J_0$ is a constant.

Given the solution (6.8) to the resolving system (6.7), we now implement the reconstruction step of the group foliation algorithm. Let $\overline{\rho}_n$ denote the left moving frame. Recall from Example 4.4 that the Maurer–Cartan invariant is $\mathfrak{m}_n = H_n = h$. Thus, the reconstruction equation $\overline{\rho}_{n+1} = \overline{\rho}_n\, \mathfrak{m}_n$ reduces to

$$\overline{\rho}_{n+1} = \overline{\rho}_n + H_n = \overline{\rho}_n + h, \tag{6.9}$$

since the one-parameter symmetry group is $G = (\mathbb{R}, +)$. The solution to the reconstruction equation (6.9) is

$$\overline{\rho}_n = h\, n + \overline{\rho}_0, \tag{6.10}$$

where $\overline{\rho}_0$ is an arbitrary constant, which reflects the automorphic property of the reconstruction equation (6.9). Indeed, two solutions to the reconstruction equation (6.9) will differ by at most an additive constant.

Since $\iota_n(x_n) = 0$ and $\iota_n(y_n) = [(b-1)J_n]^{1/(1-b)}$, the solution to the original system of finite difference equations (2.4) is obtain by acting on $(0, [(b-1)J_n]^{1/(1-b)})$ with the left moving frame (6.10). The result is

$$
\begin{aligned}
x_n &= \overline{\rho}_n \cdot 0 = h\,n + \overline{\rho}_0 \\
y_n &= \overline{\rho}_n \cdot [(b-1)J_n]^{1/(1-b)} = (1-b)^{1/(1-b)} \left[ k\,x_n + \frac{x_n^{1+a}}{1+a} + C \right]^{1/(1-b)},
\end{aligned}
$$

where $C = -J_0 - k\,\epsilon_0$ is a constant.

**Example 6.6.** As a second example, we consider the finite difference equations

$$
\frac{y_{n+1} - 2y_n + y_{n-1}}{(x_n - x_{n-1})^2} = \left( \frac{y_n - y_{n-1}}{x_n - x_{n-1}} \right)^{1/2}, \qquad \frac{(x_{n+1} - x_n)^{3/2}}{(y_{n+1} - y_n)^{1/2}} = \frac{(x_n - x_{n-1})^{3/2}}{(y_n - y_{n-1})^{1/2}}. \quad (6.11)
$$

These equations provide an invariant approximation of the second order ordinary differential equation

$$
\frac{3}{2} y_{xx} = \sqrt{y_x}
$$

on a non-uniform mesh. The equations (6.11) admit the three-dimensional solvable symmetry group

$$
X_n = \lambda\,x_n + a, \qquad Y_n = \lambda^3 y_n + b, \qquad \text{where} \qquad \lambda > 0 \quad \text{and} \quad a, b \in \mathbb{R}.
$$

We now solve (6.11) using the group foliation method. For convenience, we choose the cross-section

$$
\mathcal{K} = \{x_n = 0,\ y_n = 0,\ y_{n-1} = x_{n-1}\}. \quad (6.12)
$$

We do not compute the corresponding right moving frame $\rho_n$, as this is not necessary for the implementation of the group foliation algorithm. Let

$$
-J_n = \iota_n(y_{n-1}), \qquad K_n = \iota_n(x_{n+1}), \qquad I_n = \iota_n(y_{n+1}) \quad (6.13)
$$

denote three particular normalized joint invariants. Since the problem at hand involves two functions $(x, y)$ of one discrete variable $n \in \mathbb{Z}$, the three invariants (6.13) must be related by a syzygy. To find this relation we compute the recurrence relations (4.3). First, we note that the equation $y_{n-1} = x_{n-1}$ in the definition of the cross-section (6.12) implies that

$$
\iota_n(x_{n-1}) = \iota_n(y_{n-1}) = -J_n.
$$

To write down the recurrence relations among the normalized invariants, we then need to determine the Maurer–Cartan invariants $\mathfrak{m}_n = (\lambda_n, a_n, b_n)$. From the recurrence relation

$$
(-J_n, -J_n) = \rho_n \cdot (x_{n-1}, y_{n-1}) = \mathfrak{m}_{n-1}^{-1} \cdot (0, 0),
$$

it follows that $\mathfrak{m}_{n-1} \cdot (-J_n, -J_n) = (0, 0)$. Shifting by $\mathbf{S}^+$, we obtain the equations

$$
a_n = \lambda_n\,J_{n+1}, \qquad b_n = \lambda_n^3\,J_{n+1}. \quad (6.14)
$$

Similarly, the recurrence relation

$$(K_n,\, I_n) = \rho_n \cdot (x_{n+1}, y_{n+1}) = \mathfrak{m}_n \cdot (0,0)$$

yields

$$K_n = a_n, \qquad I_n = b_n. \tag{6.15}$$

Combining (6.14) and (6.15), we find that the Maurer–Cartan invariants are given by

$$\lambda_n = \left(\frac{I_n}{J_{n+1}}\right)^{1/3}, \qquad a_n = I_n^{1/3} J_{n+1}^{2/3}, \qquad b_n = I_n. \tag{6.16}$$

We also deduce the syzygy

$$K_n = I_n^{1/3}\, J_{n+1}^{2/3}.$$

By Theorem 5.7, a generating set for the algebra of joint invariants is given by $\{n, I_n, J_n\}$.

Now that a generating set of invariants has been identified, we express (6.11) in terms of the invariants $\{n, I_n, J_n\}$ and their shifts to obtain the resolving system. As in the previous example, there is no syzygy among the generating invariants. Once the system of equations (6.11) and been invariantized and simplified, we obtain the reduced system of equations

$$I_n = J_n + J_n^2, \qquad J_{n+1} = J_n.$$

The second equation implies that $J_n$ is constant and thereby $I_n$ also. Thus,

$$I_n = J_0 + J_0^2, \qquad J_n = J_0, \tag{6.17}$$

where $J_0$ is a constant. Substituting (6.17) in (6.16), the Maurer–Cartan invariants reduce to

$$\lambda_n = A, \qquad a_n = A\, J_0, \qquad b_n = A^3\, J_0,$$

where $A = (1 + J_0)^{1/3}$.

To compute the reconstruction equations, let $\overline{\rho}_n = (\overline{\lambda}_n, \overline{a}_n, \overline{b}_n)$ denote the components of the left moving frame. Then, the reconstruction equations $\overline{\rho}_{n+1} = \overline{\rho}_n\, \mathfrak{m}_n$ are

$$\overline{\lambda}_{n+1} = A\, \overline{\lambda}_n, \qquad \overline{a}_{n+1} = J_0 A \overline{\lambda}_n + \overline{a}_n, \qquad \overline{b}_{n+1} = J_0 (A \overline{\lambda}_n)^3 + \overline{b}_n.$$

Provided $A > 0$ (this is to guarantee that $\overline{\lambda}_n > 0$), the solution is

$$\overline{\lambda}_n = \lambda_0\, A^n, \qquad \overline{a}_n = a_0 + J_0 \lambda_0 \sum_{k=1}^{n} A^k, \qquad \overline{b}_n = b_0 + J_0 \lambda_0^3 \sum_{k=1}^{n} A^{3k}, \tag{6.18}$$

where $\lambda_0 > 0$, $a_0$, $b_0$ are integration constants.

Finally, the solution to the original system of finite difference equations (6.11) is obtained by acting on the phantom invariants $\iota_n(x_n) = \iota_n(y_n) = 0$ with the left moving frame (6.18):

$$(x_n, y_n) = \overline{\rho}_n \cdot (0,0) = \left(a_0 + J_0 \lambda_0 \sum_{k=1}^{n} A^k,\, b_0 + J_0 \lambda_0^3 \sum_{k=1}^{n} A^{3k}\right).$$

**Example 6.7.** In this example we consider the system of finite difference equations

$$
\begin{gathered}
\frac{1}{y_{m,n+1}-y_{m,n}}\left[\frac{u_{m+1,n+1}-u_{m,n+1}}{x_{m+1,n+1}-x_{m,n+1}}-\frac{u_{m+1,n}-u_{m,n}}{x_{m+1,n}-x_{m,n}}\right]=u_{m,n},\\
x_{m+1,n}-x_{m,n}=h, \qquad x_{m,n+1}-x_{m,n}=0,\\
y_{m+1,n}-y_{m,n}=0, \qquad y_{m,n+1}-y_{m,n}=k,
\end{gathered}
\tag{6.19}
$$

where $h,\, k>0$ are positive constants. We foliate (6.19) with respect to the three-dimensional symmetry group

$$
X_{m,n}=x_{m,n}+a, \qquad Y_{m,n}=y_{m,n}+b, \qquad U_{m,n}=\lambda\, u_{m,n}, \qquad \lambda\neq 0, \quad a,b\in\mathbb{R}. \tag{6.20}
$$

Assuming $u_{m,n}\neq 0$, a cross-section to the product group action (6.20) is given by

$$
\mathcal{K}=\{x_{m,n}=0,\, y_{m,n}=0,\, u_{m,n}=1\}.
$$

Let

$$
\begin{aligned}
I_{m,n}&=\iota_{m,n}(x_{m+1,n}), & J_{m,n}&=\iota_{m,n}(x_{m,n+1}), & K_{m,n}&=\iota_{m,n}(y_{m+1,n}),\\
H_{m,n}&=\iota_{m,n}(y_{m,n+1}), & V_{m,n}&=\iota_{m,n}(u_{m+1,n}), & W_{m,n}&=\iota_{m,n}(u_{m,n+1}),
\end{aligned}
\tag{6.21}
$$

denote some of the normalized invariants. By Theorem 5.4, these normalized invariants, together with $(m,n)$, form a generating set of the algebra of joint invariants. Let

$$
\begin{aligned}
\mathfrak{m}_{m,n}&=\rho_{m,n}\cdot\rho_{m+1,n}^{-1}=(a_{m,n},\, b_{m,n},\, \lambda_{m,n}),\\
\widetilde{\mathfrak{m}}_{m,n}&=\rho_{m,n}\cdot\rho_{m,n+1}^{-1}=(\widetilde{a}_{m,n},\, \widetilde{b}_{m,n},\, \widetilde{\lambda}_{m,n}),
\end{aligned}
$$

denote the Maurer–Cartan invariants. In terms of the normalized invariants (6.21), these are given by

$$
\begin{aligned}
(I_{m,n},\, K_{m,n},\, V_{m,n})&=\rho_{m,n}\cdot(x_{m+1,n},\, y_{m+1,n},\, u_{m+1,n})=\mathfrak{m}_{m,n}\cdot(0,0,1)\\
&=(a_{m,n},\, b_{m,n},\, \lambda_{m,n}),\\
(J_{m,n},\, H_{m,n},\, W_{m,n})&=\rho_{m,n}\cdot(x_{m,n+1},\, y_{m,n+1},\, u_{m,n+1})=\widetilde{\mathfrak{m}}_{m,n}\cdot(0,0,1)\\
&=(\widetilde{a}_{m,n},\, \widetilde{b}_{m,n},\, \widetilde{\lambda}_{m,n}).
\end{aligned}
$$

Unlike the previous two examples, there are now syzygies among the generating invariants (6.21). Since

$$
\rho_{m,n}\cdot(x_{m+1,n+1},\, y_{m+1,n+1},\, u_{m+1,n+1})=\mathfrak{m}_{m,n}\cdot\widetilde{\mathfrak{m}}_{m+1,n}\cdot(0,0,1)
$$

and

$$
\rho_{m,n}\cdot(x_{m+1,n+1},\, y_{m+1,n+1},\, u_{m+1,n+1})=\widetilde{\mathfrak{m}}_{m,n}\cdot\mathfrak{m}_{m,n+1}\cdot(0,0,1)
$$

we obtain the equality

$$
\mathfrak{m}_{m,n}\cdot\widetilde{\mathfrak{m}}_{m+1,n}\cdot(0,0,1)=\widetilde{\mathfrak{m}}_{m,n}\cdot\mathfrak{m}_{m,n+1}\cdot(0,0,1).
$$

Expanding reveals three fundamental syzygies:

$$
\begin{gathered}
W_{m,n}\, V_{m,n+1}=V_{m,n}\, W_{m+1,n},\\
I_{m,n+1}-J_{m+1,n}=I_{m,n}-J_{m,n}, \qquad K_{m,n+1}-H_{m+1,n}=K_{m,n}-H_{m,n}.
\end{gathered}
\tag{6.22a}
$$

Invariantizating the system of equations (6.19), and expressing the result in terms of the generating invariants (6.21), we obtain

$$\frac{1}{H_{m,n}}\left[\frac{V_{m,n}W_{m+1,n}-W_{m,n}}{I_{m,n+1}}-\frac{V_{m,n}-1}{I_{m,n}}\right]=1, \qquad I_{m,n}=h, \qquad J_{m,n}=0, \qquad K_{m,n}=0, \qquad H_{m,n}=k. \tag{6.22b}$$

After some simplifications, the resolving system (6.22) is equivalent to

$$V_{m,n}W_{m+1,n}-W_{m,n}-V_{m,n}+1=h\,k, \qquad W_{m,n}V_{m,n+1}=V_{m,n}W_{m+1,n}, \qquad I_{m,n}=h, \qquad J_{m,n}=0, \qquad K_{m,n}=0, \qquad H_{m,n}=k. \tag{6.23}$$

It follows that the invariants $I_{m,n}$, $J_{m,n}$, $K_{m,n}$ and $H_{m,n}$ are constant.

Assuming a solution for $V_{m,n}$ and $W_{m,n}$ is known, we now proceed to the reconstruction procedure. Let $\overline{\rho}_{m,n}=(\overline{a}_{m,n},\,\overline{b}_{m,n},\,\overline{\lambda}_{m,n})$ denote the left moving frame. Writing down each component of the reconstruction equations

$$\overline{\rho}_{m+1,n}=\overline{\rho}_{m,n}\,\mathfrak{m}_{m,n}, \qquad \overline{\rho}_{m,n+1}=\overline{\rho}_{m,n}\,\widetilde{\mathfrak{m}}_{m,n},$$

we obtain the equations

$$\begin{aligned} \overline{\lambda}_{m+1,n}&=\overline{\lambda}_{m,n}V_{m,n}, & \overline{\lambda}_{m,n+1}&=\overline{\lambda}_{m,n}W_{m,n},\\ \overline{a}_{m+1,n}&=\overline{a}_{m,n}+h, & \overline{a}_{m,n+1}&=\overline{a}_{m,n},\\ \overline{b}_{m+1,n}&=\overline{b}_{m,n}, & \overline{b}_{m,n+1}&=\overline{b}_{m,n}+k. \end{aligned} \tag{6.24}$$

The solutions for $\overline{a}_{m,n}$ and $\overline{b}_{m,n}$ are

$$\overline{a}_{m,n}=a_0+h\,m, \qquad \overline{b}_{m,n}=b_0+k\,n,$$

where $a_0$ and $b_0$ are constants. The solution for $\overline{\lambda}_{m,n}$ will depend on $V_{m,n}$ and $W_{m,n}$. Assuming $\overline{\lambda}_{m,n}$ is known, a solution to the original system of equations (6.19) is given by

$$(x_{m,n},y_{m,n},u_{m,n})=\overline{\rho}_{m,n}\cdot(0,0,1)=(a_0+h\,m,b_0+k\,n,\overline{\lambda}_{m,n}),$$

The first two equations in the resolving system (6.23) are nonlinear finite difference equations in $V_{m,n}$ and $W_{m,n}$. Particular solutions can be obtained by making simplifying assumptions about the form of the solution. For example, assume $W_{m,n}=W\neq 1$ is constant. Then, the resolving system (6.23) yields

$$V_{m,n}=1+\frac{h\,k}{W-1}.$$

Solving the reconstruction equation (6.24) for $\overline{\lambda}_{m,n}$, we obtain

$$\overline{\lambda}_{m,n}=\lambda_0\,W^n\left(1+\frac{h\,k}{W-1}\right)^m, \qquad \text{where} \qquad \lambda_0\neq 0.$$

Thus,

$$x_{m,n}=a_0+h\,m, \qquad y_{m,n}=b_0+k\,n, \qquad u_{m,n}=\lambda_0\,W^n\left(1+\frac{h\,k}{W-1}\right)^m$$

is a particular solution of (6.7).

# 7 A discrete Schwarz's Theorem

Consider the Schwarzian equation

$$\frac{y_{xxx}}{y_x} - \frac{3}{2}\left(\frac{y_{xx}}{y_x}\right)^2 = F(x), \tag{7.1}$$

which is invariant under the group of linear fractional transformations

$$X = x, \qquad Y = \frac{ay+b}{cy+d}, \qquad ad - bc = 1.$$

Schwarz's Theorem, [13, 23], states that the general solution of (7.1) is given by the ratio

$$y(x) = \frac{\psi_1(x)}{\psi_2(x)},$$

where $\psi_1(x)$ and $\psi_2(x)$ are linearly independent solutions of the linear equation

$$\psi_{xx} + \frac{1}{2}F(x)\psi = 0.$$

As an application of the group foliation method, we derive an analogous theorem in the discrete setting, expressing the solution of an invariant discretization of the Schwarz equation as a ratio of solutions to a linear finite difference equation.

For an invariant discretization of (7.1), we take the finite difference equations

$$x_{n+1} - x_n = h, \qquad \frac{6}{h^2} - \frac{8}{h^2}\frac{(y_{n+2}-y_{n+1})(y_{n+3}-y_n)}{(y_{n+2}-y_n)(y_{n+3}-y_{n+1})} = F(x_n), \tag{7.2}$$

where $h$ is a constant. These equations are invariant under the product action

$$X_n = x_n, \qquad Y_n = \frac{ay_n+b}{cy_n+d}, \qquad ad - bc = 1.$$

We now apply the group foliation method to (7.2). Using the cross-section introduced in [31], we let

$$\mathcal{K} = \{y_n = 0,\ y_{n+1} \to \infty,\ y_{n+2} = \epsilon_n\},$$

where

$$\epsilon_n = \operatorname{sign}\left(-\frac{y_{n+2}-y_n}{(y_{n+1}-y_n)(y_{n+2}-y_{n+1})}\right).$$

To avoid some complications, we restrict ourselves to the case where $\epsilon_n = \epsilon$ is constant. A generating set of the algebra of joint invariants is given by the index $n$ and the invariants

$$x_n, \qquad I_n = \epsilon \cdot \iota_n(y_{n+3}) = \frac{(y_{n+2}-y_{n+1})(y_{n+3}-y_n)}{(y_{n+2}-y_n)(y_{n+3}-y_{n+1})}.$$

There is no syzygy among $x_n$ and $I_n$. Therefore, by invariantizing (7.2), we obtain the full resolving system

$$x_{n+1} - x_n = h, \qquad 6 - 8I_n = h^2 F(x_n).$$

The first equation is easily solved to obtain

$$x_n = hn + x_0,$$

and the second equation uniquely specifies $I_n$ in terms of the function $F(x_n)$:

$$I_n = \frac{3}{4} - \frac{h^2}{8}F(x_n) = \frac{3}{4} - \frac{h^2}{8}F(hn + x_0). \tag{7.3}$$

We now compute the corresponding reconstruction equations. To do so, we first need to determine symbolic expressions for the Maurer–Cartan invariants

$$\mathfrak{m}_n = \begin{bmatrix} a_n & b_n \\ c_n & d_n \end{bmatrix} \in SL(2, \mathbb{R}).$$

Using the recurrence relations (4.3), we first have, in the limit,

$$\infty \leftarrow \iota_n(y_{n+1}) = \mathfrak{m}_n \cdot \iota_{n+1}(y_{n+1}) = \mathfrak{m}_n \cdot 0 = \frac{b_n}{d_n}.$$

Requiring that the Maurer–Cartan invariants be finite, set

$$d_n = 0.$$

Next, we have the recurrence relations

$$\epsilon = \iota_n(y_{n+2}) = \mathfrak{m}_n \cdot \iota_{n+1}(y_{n+2}) \to \mathfrak{m}_n \cdot \infty = \frac{a_n}{c_n}$$

and

$$\iota_n(y_{n+3}) = \mathfrak{m}_n \cdot \iota_{n+1}(y_{n+3}) = \mathfrak{m}_n \cdot \epsilon = \epsilon + \frac{b_n}{c_n}\epsilon.$$

Using the unitary constraint $a_n d_n - b_n c_n = 1$, we obtain

$$b_n = -\epsilon\sqrt{|1 - I_n|} \neq 0$$

and

$$c_n = \frac{\epsilon}{\sqrt{|1 - I_n|}}, \qquad a_n = \frac{1}{\sqrt{|1 - I_n|}}.$$

Therefore, in matrix form, the Maurer–Cartan invariants are

$$\mathfrak{m}_n = \begin{bmatrix} \Delta_n & -\dfrac{\epsilon}{\Delta_n} \\ \epsilon\,\Delta_n & 0 \end{bmatrix}, \qquad \text{where} \qquad \Delta_n = \frac{1}{\sqrt{|1 - I_n|}}.$$

Taking into account that $I_n$ is given by (7.3), we have

$$\Delta_n = \frac{2}{\sqrt{\left|1 + \dfrac{h^2}{2}F(hn + x_0)\right|}}. \tag{7.4}$$

Now let

$$\overline{\rho}_n = \begin{bmatrix} \overline{a}_n & \overline{b}_n \\ \overline{c}_n & \overline{d}_n \end{bmatrix}, \qquad \overline{a}_n\overline{d}_n - \overline{b}_n\overline{c}_n = 1,$$

be a left moving frame. The reconstruction equation $\overline{\rho}_{n+1} = \overline{\rho}_n \, \mathfrak{m}_n$ yields the system of equations

$$\overline{a}_{n+1} = \Delta_n(\overline{a}_n + \epsilon\, \overline{b}_n), \qquad \overline{b}_{n+1} = -\frac{\epsilon}{\Delta_n}\overline{a}_n,$$
$$\overline{c}_{n+1} = \Delta_n(\overline{c}_n + \epsilon\, \overline{d}_n), \qquad \overline{d}_{n+1} = -\frac{\epsilon}{\Delta_n}\overline{d}_n.$$

Manipulating these equations, we find that the group components $\overline{b}_n$, $\overline{d}_n$ must satisfy the second order linear finite difference equation

$$\beta_{n+2} + \frac{\Delta_n}{\Delta_{n+1}}(-\Delta_n \beta_{n+1} + \beta_n) = 0. \tag{7.5}$$

Acting on the normalized invariant $\iota_n(y_n) = 0$ by the left moving frame $\overline{\rho}_n$, we conclude that a solution to (7.2) is given by

$$x_n = hn + x_0, \qquad y_n = \frac{\overline{b}_n}{\overline{d}_n}, \tag{7.6}$$

where $\overline{b}_n$ and $\overline{d}_n$ are two linearly independent solutions of (7.5). Thus, by applying the group foliation method to (7.2), we have deduced a discrete analog of Schwarz's Theorem.

**Theorem 7.1.** The general solution to the system of finite difference equations

$$x_{n+1} - x_n = h, \qquad \frac{6}{h^2} - \frac{8}{h^2}\frac{(y_{n+2} - y_{n+1})(y_{n+3} - y_n)}{(y_{n+2} - y_n)(y_{n+3} - y_{n+1})} = F(x_n),$$

is given by

$$x_n = hn + x_0, \qquad y_n = \frac{\overline{b}_n}{\overline{d}_n},$$

where $\overline{b}_n$ and $\overline{d}_n$ are linearly independent solutions to the linear finite difference equation

$$\beta_{n+2} + \frac{\Delta_n}{\Delta_{n+1}}(-\Delta_n \beta_{n+1} + \beta_n) = 0,$$

with $\Delta_n$ given in (7.4).

To conclude this investigation more concretely, we examine the group foliation formulation (7.6) of (7.2) for some specific functions $F(x_n)$. First we consider $F(x_n) = 0$. In this case $\Delta_n = 2$ and (7.5) simplifies to

$$\beta_{n+2} - 2\beta_{n+1} + \beta_n = 0.$$

The general solution to this equation is

$$\beta_n = kn + \ell,$$

where $k$ and $\ell$ are constants. Using (7.6), we then find that the general solution to (7.2) when $F(x_n) = 0$ is

$$x_n = hn + x_0, \qquad y_n = \frac{an + b}{cn + d} = \frac{Ax_n + B}{Cx_n + D}, \tag{7.7}$$

with $AD - BC \neq 0$. In this particular situation, the discrete solution (7.7) agrees exactly with the solution of the differential equation (7.1).

Next we consider $F(x_n) = F > 0$, a positive constant. In this case,

$$\Delta_n = \frac{2}{\sqrt{1 + \dfrac{h^2 F}{2}}} = \Delta > 0$$

is a positive constant and (7.5) reduces to

$$\beta_{n+2} - \Delta\beta_{n+1} + \beta_n = 0.$$

The general solution to this equation is

$$\beta_n = A\cos(n\theta) + B\sin(n\theta),$$

with

$$\theta = \arctan\left(h\sqrt{\frac{F}{2}}\right). \tag{7.8}$$

Using (7.6), we find that the general solution to (7.2) when $F(x_n) = F > 0$ is

$$x_n = hn + x_0, \qquad y_n = \frac{A\cos(n\theta) + B\sin(n\theta)}{C\cos(n\theta) + D\sin(n\theta)}, \quad \text{with} \quad AD - BC \neq 0. \tag{7.9}$$

Analogously, the general solution to the Schwarzian equation (7.1) is

$$y = \frac{A\cos(\omega x) + B\sin(\omega x)}{C\cos(\omega x) + D\sin(\omega x)}, \qquad \text{where} \qquad \omega = \sqrt{\frac{F}{2}}. \tag{7.10}$$

Taking the Maclaurin expansion of (7.8), we have

$$\theta = h\sqrt{\frac{F}{2}} + O(h^3),$$

and since

$$n\theta = nh\sqrt{\frac{F}{2}} + O(h^3) = (x_n - x_0)\sqrt{\frac{F}{2}} + O(h^3),$$

we see that (7.9) is an order $h^2$ approximation of the solution (7.10).

# 8 Numerical simulation

It has been observed via numerical simulations that symmetry preserving numerical schemes can sometimes outperform traditional numerical integrators, [35, 41]. For ordinary differential equations, [2–4, 7, 15, 37], invariant schemes tend to deliver marked improvements when solutions exhibit sharp variations or singularities. Reasons for these improvements are not well understood. In this section, we use the alternative numerical scheme obtained in the previous section through group foliation to give some explanation of the improvements seen in invariant schemes. Note that our observations here are exploratory and further investigation of connections between group foliation and numerical methods still needs to be carried out.

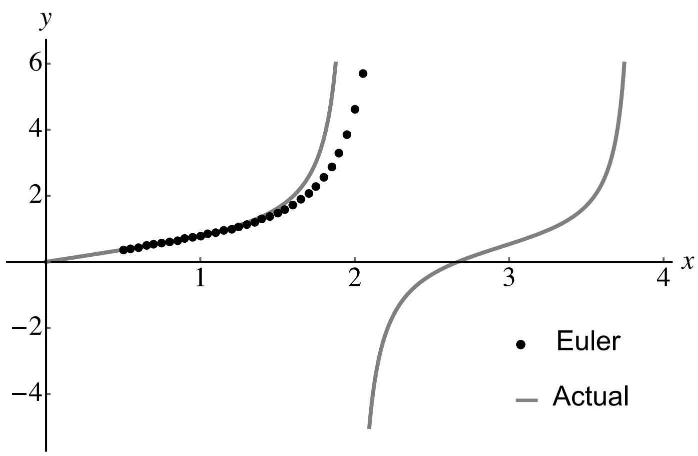
Consider the Schwarzian equation (7.1) of the previous section with $F(x) = 2x$. In this case the general solution to (7.1) is given by

$$y(x) = \frac{AJ_{-1/3}(\frac{2}{3}x^{3/2}) + BJ_{1/3}(\frac{2}{3}x^{3/2})}{CJ_{-1/3}(\frac{2}{3}x^{3/2}) + DJ_{1/3}(\frac{2}{3}x^{3/2})}, \tag{8.1}$$

where $J_\alpha(x)$ denotes a Bessel function of the first kind. Consider the particular solution

$$y(x) = \frac{J_{1/3}(\frac{2}{3}x^{3/2})}{J_{-1/3}(\frac{2}{3}x^{3/2})}, \tag{8.2}$$

corresponding to choices of $A = 0$, $B = C = 1$ and $D = 0$ in the general solution (8.1). Note that this corresponds approximately to the initial condition $y(.5) = 0.3684$, $y'(.5) = 0.7602$, $y''(.5) = 0.1925$. This solution has singularities where $J_{-1/3}(\frac{2}{3}x^{3/2}) = 0$, e.g. $x \approx 1.986, 3.825, 5.296$, et cetera.

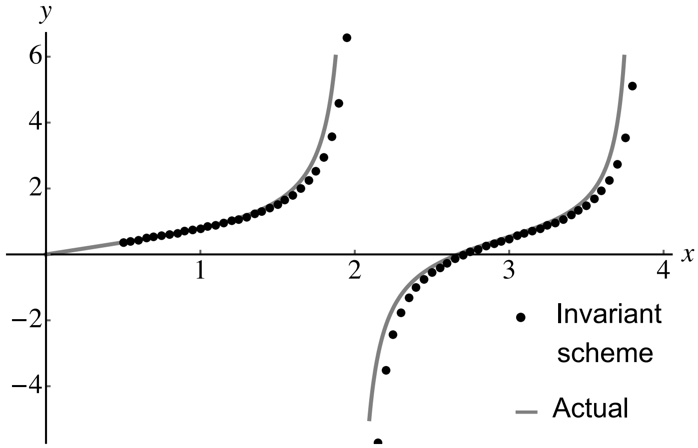



| $x$ | Actual | Euler |
|---|---|---|
| 1.8 | 3.744 | 2.543 |
| 1.85 | 4.954 | 2.873 |
| 1.9 | 7.544 | 3.297 |
| 1.95 | 17.202 | 3.857 |
| 2. | −43.764 | 4.622 |
| 2.05 | −8.912 | 5.711 |
| 2.1 | −4.708 | 7.337 |
| 2.15 | −3.060 | 9.907 |
| 2.2 | −2.173 | 14.262 |

Figure 2: Integration over an asymptote using Euler's method.

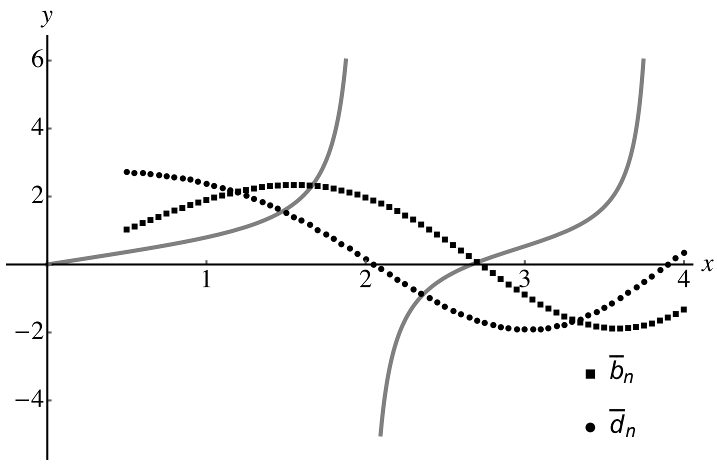
Using standard finite difference methods such as Euler or Runge–Kutta, one cannot typically integrate over an asymptote. To illustrate this concretely, we compare the particular solution (8.2) with the result of numerical integration via a fourth order Euler's method with a mesh size of $h = .05$, shown in Figure 2. As expected, Euler's method cannot track the solution across the singularity at $x \approx 1.986$.

By contrast, the invariant discretization (7.2) will track the solution across the singularities. Shown in Figure 3 is the result of integrating via the invariant scheme (7.2) with a mesh size of $h = .05$. Reasons for this behavior are not apparent from (7.2). If we instead examine the group foliation scheme (7.6), this numerical behavior becomes more transparent. As seen in Theorem 7.1, group foliation integrates a ratio $\bar{b}_n/\bar{d}_n$, where $\bar{b}_n, \bar{d}_n$ satisfy (7.5), instead of integrating $y_n$ directly. Thus the integration may pass over a singularity of $y_n$ by simply allowing $\bar{d}_n$ to pass over 0, and the functions $\bar{b}_n, \bar{d}_n$ remain bounded throughout the integration, as seen in Figure 4. Taking the ratio $\bar{b}_n/\bar{d}_n$ will reproduce exactly the values produced by the original invariant scheme (7.2).




| $x$ | Actual | (7.2) |
|---|---|---|
| 1.8 | 3.744 | 2.959 |
| 1.85 | 4.954 | 3.579 |
| 1.9 | 7.544 | 4.591 |
| 1.95 | 17.202 | 6.568 |
| 2. | −43.764 | 12.230 |
| 2.05 | −8.912 | 200.57 |
| 2.1 | −4.708 | −12.526 |
| 2.15 | −3.060 | −5.713 |
| 2.2 | −2.173 | −3.524 |

Figure 3: Integration over an asymptote using invariant scheme (7.2).




| $x$ | $\bar{b}_n$ | $\bar{d}_n$ |
|---|---|---|
| 1.8 | 2.208 | 0.7461 |
| 1.85 | 2.158 | 0.6027 |
| 1.9 | 2.098 | 0.4570 |
| 1.95 | 2.029 | 0.3090 |
| 2. | 1.951 | 0.1595 |
| 2.05 | 1.863 | 0.0093 |
| 2.1 | 1.766 | −0.1410 |
| 2.15 | 1.660 | −0.2906 |
| 2.2 | 1.546 | −0.4387 |

Figure 4: Integration over an asymptote using the group foliation scheme (7.6).

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