An arithmetic dynamical Mordell-Lang conjecture

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August 15, 2015

Silvermania!

Warmup: squares in polynomial orbits

For a field $K, f \in K(x)$, and $\alpha \in K$, the orbit $O_f(\alpha)$ is $\{f^n(\alpha) : n \geq 0\}$.

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Let $f \in \mathbb{Q}[x]$ be monic and quadratic, and let S be the set of rational squares. Suppose there is $\alpha \in \mathbb{Q}$ such that $O_f(\alpha) \cap S$ is infinite. What can be said about f?

Motivation:

- If $f \in \mathbb{Q}(x)$ has degree at least two and there is $\alpha \in \mathbb{Q}$ with $O_f(\alpha) \cap \mathbb{Z}$ infinite, then $f^2(x) \in \mathbb{Q}[x]$ (Silverman 1993)
- If $f, g \in \mathbb{C}[x]$ have degree at least two and there are $\alpha, \beta \in \mathbb{C}$ with $O_f(\alpha) \cap O_g(\beta)$ infinite, then f and g have a common iterate (Ghioca-Tucker-Zieve 2008)

Theorem (Cahn-RJ-Spear 2015)

If $f \in \mathbb{Q}[x]$ is monic and quadratic and $O_f(\alpha) \cap S$ is infinite for some $\alpha \in \mathbb{Q}$, then either

- $f(x) = (x+c)^2$ *for some* $c \in \mathbb{Q}$, *or*
- $f(x) = x^2 + 4x$.

Remarks (let $f(x) = x^2 + 4x$):

- $O_f(1/2) = \{1/2, (3/2)^2, (15/4)^2, (255/16)^2, \ldots\}$
- $f^2(x) = (x^2 + 4x)(x+2)^2$
- $f(x) = T_2(x+2) - 2$, where $T_2(x) = x^2 - 2$. Critical orbit of $f(x)$ is $-2 \mapsto -4 \mapsto 0 \mapsto 0$.
- For any monic, quadratic $f \in \mathbb{Q}[x]$ and any $\alpha \in \mathbb{Q}$, $\{n : f^n(\alpha) \in S\}$ is a finite union of arithmetic progressions.

The Dynamical Mordell-Lang conjecture

Conjecture (Dynamical Mordell-Lang)

Let $X/\mathbb{C}$ be a quasi-projective variety, $V \subseteq X$ a subvariety, and $f : X \to X$ a morphism. Then for all $\alpha \in X(\mathbb{C})$, the set $\{n : f^n(\alpha) \in V(\mathbb{C})\}$ is a finite union of arithmetic progressions.

Singletons are considered arithmetic progressions. So if $\{n : f^n(\alpha) \in V(\mathbb{C})\}$ is finite, then the conjecture holds.

Theorem (Skolem-Mahler-Lech)

If $F(x_0, \ldots, x_{\ell-1}) = \sum_{i=0}^{\ell-1} a_i x_i$ *is a linear form on* $\mathbb{C}^\ell$ *and* $a_{n+\ell} = F(a_n, \ldots, a_{n+\ell-1})$ *for all* $n \geq 0$*, then* $\{n : a_n = 0\}$ *is a finite union of arithmetic progressions.*

Special case of dynamical M-L conjecture: $f : \mathbb{A}^\ell \to \mathbb{A}^\ell$, $f(x_0, \ldots, x_{\ell-1}) = (x_1, \ldots, x_{\ell-1}, F(x_0, \ldots, x_{\ell-1}))$, $V = \{x_0 = 0\}$.

The dynamical M-L conjecture is known to hold for

- $X = \mathbb{A}^n$ and f an automorphism of X (Bell 2006)
- X a semi-abelian variety (Ghioca-Tucker 2009).
- X arbitrary and f étale (Bell-Ghioca-Tucker 2010)
- $X = \mathbb{A}^2$ (Xie 2015)
- $X = \mathbb{A}^n$, V is a curve, and $f = (f_1, \ldots, f_n)$ with $f_i \in \mathbb{C}[x]$ (Xie 2015)

A question over number fields

From now on, $\boxed{K \text{ is a number field.}}$

A K-endomorphism of a variety X is a morphism $X \to X$ defined over K.

Question: Let X/K be a quasi-projective variety, $V \subset X(K)$ the value set $\lambda(X(K))$ of a K-endomorphism λ of X, and f a K-endomorphism of X. For $\alpha \in X(K)$, must $\{n : f^n(\alpha) \in V\}$ be a finite union of arithmetic progressions?

Proposition

Let G be a finitely generated abelian group, $H \leq G$, and $f : G \to G$ a homomorphism. Then for any $\alpha \in G$, $\{n : f^n(\alpha) \in H\}$ is a finite union of arithmetic progressions.

Consequence: if X is an abelian variety, f and λ are isogenies on X, and $\alpha \in X(K)$, then $\{n : f^n(\alpha) \in \lambda(X(K))\}$ is a finite union of arithmetic progressions.

Bad example: $K = \mathbb{Q}$, $X = \mathbb{A}^1$, $\lambda(y) = y^2$, $V = \{\text{squares in } \mathbb{Q}\}$, $f(x) = x + 1$, $\alpha = 0$.

Then $f^n(0) = n$ for all $n \geq 0$, so

$$\{n : f^n(0) \in V\} = \{0, 1, 4, 9, \ldots\}.$$

A heuristic

Revised Question: Let X/K be a quasi-projective variety, λ a K-endomorphism of X, $V = \lambda(X(K))$, and f a sufficiently complicated K-endomorphism of X. For $\alpha \in X(K)$, must $\{n : f^n(\alpha) \in V\}$ be a finite union of arithmetic progressions?

Suppose there is i with $f^i = \lambda \circ g$, where g is a K-endomorphism of X.
Then for $n \geq i$, we have $f^n(\alpha) = \lambda(g(f^{n-i}(\alpha))) \in \lambda(X(K))$.

So if an iterate of f has a "close functional relationship" to λ, we should expect the question to have an affirmative answer.

For $n \geq 1$, let Z_n be the subvariety of $X \times X$ given by $f^n(x) = \lambda(y)$.

Then there is a natural K-morphism $f : Z_{n+1} \to Z_n$ taking (x, y) to $(f(x), y)$. Thus if $i > j$, a point in $Z_i(K)$ maps to a point in $Z_j(K)$.

Suppose that $\{n : f^n(\alpha) \in \lambda(X(K))\}$ is infinite.

Then $Z_n(K)$ is infinite for all $n \geq 1$.

First leap of faith: For each n, the infinitely many points in $Z_n(K)$ are Zariski dense in Z_n.

Second leap of faith: The Bombieri-Lang conjecture is true: if a variety has a Zariski-dense set of K-rational points, then it is not of general type (i.e. not of full Kodaira dimension). Therefore Z_n is not of general type for any n.

Third leap of faith: Because f is sufficiently complicated, the varieties Z_n will be of general type for large n unless some iterate of f has a "close functional relationship" to λ.

Conjecture (Arithmetic dynamical Mordell-Lang conjecture)

Let $X = (\mathbb{P}^1)^g$ and let $f = (f_1, \ldots, f_g)$ with $f_i \in K(x)$, $\deg f_i \geq 2$. Then for any K-endomorphism λ of X and any $\alpha \in X(K)$, the set $\{n : f^n(\alpha) \in \lambda(X(K))\}$ is a finite union of arithmetic progressions.

If $\lambda = (\lambda_1, \ldots, \lambda_g)$ with $\lambda_i \in K(x)$, then the conjecture may be proved one coordinate at a time, and reduces to the case where $X = \mathbb{P}^1$.

Theorem (Cahn-RJ-Spear)

The conjecture holds for $X = \mathbb{P}^1$ *and* $\lambda(y) = y^m$, *where* $m \in \mathbb{Z}$.

Proof Sketch

Let $f \in K(x)$, and note Z_n is the curve $f^n(x) = y^m$. Suppose that $O_f(\alpha) \cap (\mathbb{P}^1(K))^m$ is infinite, so that $Z_n(K)$ is infinite for each n.

~~**First leap of faith**~~ **First fact:** For each n, the infinitely many points in $Z_n(K)$ are Zariski-dense in Z_n.

~~**Second leap of faith**~~ **Second fact:** The Bombieri-Lang conjecture is true for curves (Faltings' Theorem). Therefore Z_n is not of general type for any n, i.e. the genus of Z_n is ≤ 1.

~~**Third leap of faith**~~ **Third step**: Show the genus of $Z_n : f^n(x) = y^m$ is at least two unless some iterate of f has a "close functional relationship" to λ.

Definition

For $\beta \in \mathbb{P}^1(\mathbb{C})$, define $\rho_n(\beta)$ to be the number of $z \in f^{-n}(\beta)$ with $e_{f^n}(z)$ not divisible by m. Call β m-**branch abundant** for f if $\rho_n(\beta)$ is bounded as $n \to \infty$.

From genus formulae for superelliptic curves, the genus of Z_n is bounded if and only if 0 and ∞ are m-branch abundant for f.

We classified all rational functions over $\mathbb{C}$ with two m-branch abundant points, and showed their components are defined over K.

First attempt: determine all possible ramification structures of pre-image trees of an m-branch abundant point.

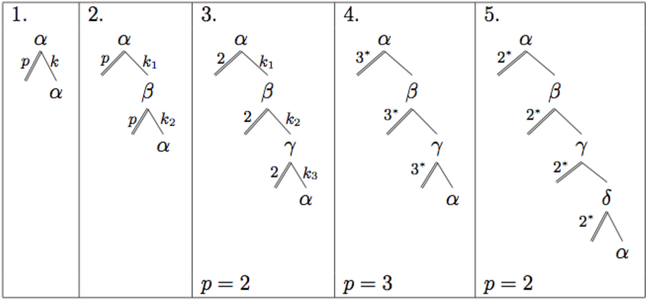

FIGURE 1. Ramification structures for $O^{-}(\alpha)$, where α is p-branch abundant for $f \in \mathbb{C}(z)$ and $p \nmid \deg f$.

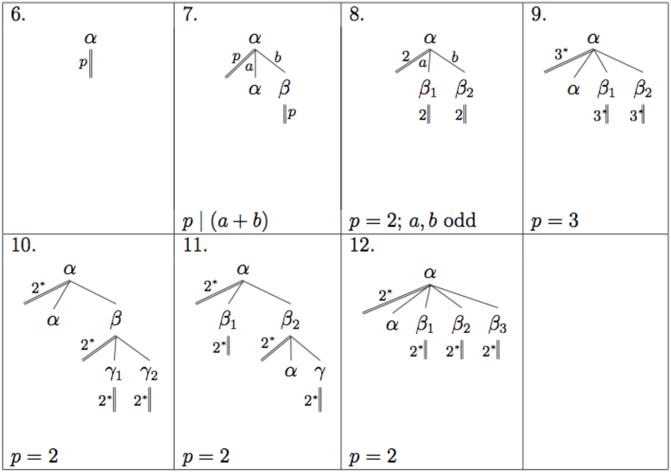

FIGURE 2. Ramification structures for $O^{-}(\alpha)$, where α is p-branch abundant for $f \in \mathbb{C}(z)$ and $p \mid \deg f$.

Theorem (Cahn-RJ-Spear)

Let $f \in K(x)$ *and fix* $m \geq 2$. *Then the genus of* $Z_n : f^n(x) = y^m$ *is bounded as* $n \to \infty$ *if and only if one of the following holds:*

- $f(x) = cx^j(g(x))^m$ *with* $g(x) \in K(x)$, $0 \leq j \leq m-1$, $c \in K^*$;
- *(requires* $m \in \{2,3,4\}$*)* f *is a Lattès map with* 0 *and* ∞ *in its post-critical set;*
- *(requires* $m = 2$*) Either* $f(x)$ *or* $1/f(1/x)$ *can be written in one of the following ways* ($B, C \in K^*, p, q, r \in K[x] \setminus \{0\}$)*:*
    1. $-\frac{p(x)^2}{(x-C)q(x)^2}$ *with* $p(x)^2 + C(x-C)q(x)^2 = Cxr(x)^2$;
    2. $-\frac{(x-C)p(x)^2}{q(x)^2}$ *with* $(x-C)p(x)^2 + Cq(x)^2 = xr(x)^2$;
    3. $B\frac{(x-C)p(x)^2}{q(x)^2}$ *with* $B(x-C)p(x)^2 - Cq(x)^2 = -Cr(x)^2$;
    4. $B\frac{x(x-C)p(x)^2}{q(x)^2}$ *with* $Bx(x-C)p(x)^2 - Cq(x)^2 = -Cr(x)^2$;

In each case of the theorem, the genus of Z_n is at most 1 for all n.

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Lattès maps

We say $f \in \mathbb{C}(z)$ is a *Lattès map* if there is an elliptic curve E, a morphism $\mu : E \to E$, and a finite separable map π such that the following diagram commutes:

$$\begin{tikzcd}
E \arrow[r, "\mu"] \arrow[d, "\pi"'] & E \arrow[d, "\pi"] \\
\mathbb{P}^1 \arrow[r, "f"'] & \mathbb{P}^1
\end{tikzcd}$$

Natural choices: π is the x-coordinate projection and $\mu = [j]$.

Question

Let $X = \mathbb{A}^2$ and $\lambda(y_1, y_2) = (y_1^{m_1}, y_2^{m_2})$ with $m_1, m_2 \geq 2$. Are there interesting examples of $f : \mathbb{A}^2 \to \mathbb{A}^2$ not of the form $(f_1(x_1), f_2(x_2))$ such that $Z_n : f^n(x_1, x_2) = (y_1^{m_1}, y_2^{m_2})$ is a surface of Kodaira dimension < 2 for all n?

Corollary

Let $f \in K(x)$, fix $m \geq 2$, and suppose that the genus of Z_n is bounded as $n \to \infty$. Then there exist $a > b \geq 0$ with $f^a(x) = f^b(x)(g(x))^m$ for some $g(x) \in K(x)$.

Corollary

$\{n : f^n(\alpha) \in (\mathbb{P}^1(K))^m\}$ is a finite union of arithmetic progressions, of modulus bounded by $a - b$.

Maximum modulus?

Example: let

$$f(x) = \frac{2(x-2)(x+2)^3}{x(x-4)^3}.$$

Then $a = 3, b = 0$ ($f^3(x) = x(g(x))^3$), and no smaller a, b suffice.

$$O_f(6) = \left\{6, \frac{4}{3} \cdot 4^3, \left(\frac{655}{488}\right)^3, 6\left(-\frac{129900299507}{120418942015}\right)^3, \ldots\right\}$$

Indeed, for all $m \geq 3$ the modulus is bounded by m, and this is best possible (independent of K):

Let $f(x) = cx(x+1)^m$, where $c \notin K^p$ for each prime p dividing m.

Then $f^i(1) = c^i(k_i)^m$ for $k_i \in K$, for all $1 \leq i \leq m-1$. But $c^i \notin K^m$, and so $\{n : f^n(1) \in (\mathbb{P}^1(K))^m\} = \{0, m, 2m, 3m, \ldots\}$.

For $m = 2$ one must have $a - b \leq 4$. This is attained by certain Lattès maps descending from CM elliptic curves.

Example:

$$f(x) = (8 + 4\sqrt{3})\frac{(x-1)(x-(4+4\sqrt{3}))^2}{x(x-(6+4\sqrt{3}))^2}$$

has post-critical orbit

$$0 \to \infty \to 8 + 4\sqrt{3} \to 1 \to 0.$$

Thus $f^4(x) = x(g(x))^4$, but $f^i(x)$ is not of this form for $i = 1, 2, 3$.

This map arises from taking E to have CM by $\mathbb{Z}[\sqrt{-3}]$, $\mu(P) = [\sqrt{-3}]P + T$, where T is a non-trivial 2-torsion point, and π to be projection onto the x-coordinate.

Question 1: Is it possible for a Lattès map with a post-critical four-cycle to have $\alpha \in K$ with $\{n : f^n(\alpha) \in (\mathbb{P}^1(K))^2\}$ an arithmetic progression of modulus 4?

Question 2: Can Lattès maps with a post-critical four-cycle be defined over $\mathbb{Q}$?

Thank you!