Achievement Sets of Sequences

Rafe Jones

July 21, 2010

Abstract

Given a real sequence (x_n), we examine the set of all sums of the form $\sum_{i\in I} x_i$, as I varies over subsets of the natural numbers. We call this the achievement set of (x_n), and write it $AS(x_n)$. For instance, $AS(1/2^n) = [0,1]$ by the existence of binary expansions, and $AS(2/3^n)$ is the Cantor middle third set. We explore the properties of these two sequences that account for their very different achievement sets. We give a sufficient condition for a sequence to have an achievement set that is an interval, and another sufficient condition for the achievement set to be a Cantor set. We also examine what sets can occur as achievement sets, and give results on the topology of achievement sets.

Introduction.

In 1854, Bernhard Riemann proved his well-known rearrangement theorem, which states that the terms of a conditionally convergent series may be rearranged so that the series sums to any specified real number (or $\pm\infty$). On the other hand, rearrangements of terms of an absolutely convergent series have no effect on the sum. In this paper, we consider a variant of the rearrangement problem: what if we allow omissions of terms (and not rearrangements)? More precisely, we say $r \in \mathbb{R}$ is *achieved by* a real sequence (x_n) if there is a (possibly finite) subsequence of (x_n) whose sum converges to r. We seek to understand all r that are achieved by a given (x_n), and we call this set the *achievement set* of (x_n), denoted $AS(x_n)$.

Two examples motivate our explorations. First, consider that the existence of binary expansions shows that $AS(1/2^n) = [0,1]$. On the other hand, $AS(2/3^n)$ is the Cantor middle third set, since the latter consists of those numbers in the unit interval representable by a ternary expansion consisting only of the digits 0 and 2. The vast topological differences between these sets prompt natural questions. What properties of these sequences make their achievement sets so different? What other sets can occur as achievement sets? In this paper we resolve the first question, and shed some light on the second.

In the direction of the first question, we characterize in Section 1 the (x_n) with limit zero such that $AS(x_n)$ is an interval, and deduce several corollaries. One of them is an analogue of Riemann's rearrangment theorem: if the terms of

(x_n) form a conditionally convergent series, then $AS(x_n) = \mathbb{R}$. In Section 2 we give a condition that ensures $AS(x_n)$ is a Cantor set, which requires proving that $AS(x_n)$ is closed provided (x_n) approaches zero. Towards the second question mentioned in the previous paragraph, we show in Section 3 that achievement sets come in two distinct flavors: with empty interior, or with dense interior. Moreover, we give conditions on (x_n) that imply $AS(x_n)$ is either a finite union of intervals or a finite union of Cantor sets. In Section 4, we give some examples of classes of sets that do occur as achievement sets, and on the other hand show that many familiar sets do not. Curiously, we find that the set of nonnegative rational numbers $\mathbb{Q}^+$ is in the latter category, while $\{-1\} \cup \mathbb{Q}^+$ is in the former.

Various authors have investigated aspects of achievement sets. Hornich [5], Kakeya [6], and Ribenboim [10, Chapter 2] have results in the direction of those presented in Section 1. Hornich [5] and Morán [7] have work along the lines of that presented in Section 2. The results of Sections 3 and Section 4 appear to be new. In [7, 8], Morán allows sequences consisting of of vectors in $\mathbb{R}^n$, and examines the case where the Lebesgue measure of the resulting achievement set is zero. He gives precise results on the Hausdorff dimension of such achievement sets, particularly in the case of sequences satisfying the conditions of Theorem 2.1. Also related are [2] and [4], in which a sequence of positive integers is called *complete* if every natural number is the sum of some subsequence. In [2], J. L. Brown showed that the Fibonacci sequence is complete, but if any two terms are removed the resulting sequence is not complete.

1 Intervals.

Throughout, we deal only with sequences whose terms are all nonzero. Moreover, all sequences are infinite unless explicitly noted otherwise. We denote a sequence $x_1, x_2, x_3, \ldots$ by (x_n), and we declare that the empty subsequence sums to 0.

For our first results on achievement sets, we give conditions on (x_n) that imply that $AS(x_n)$ is an interval. In keeping with the terminology introduced so far, we call (x_n) a *high achiever* if $AS(x_n)$ is an interval (we refrain from calling (x_n) *remedial* if $AS(x_n)$ fails to be an interval). The notion of a high achiever is the analogue of a complete sequence of positive integers, since it requires $AS(x_n)$ to be as large as possible. The following theorem gives a characterization of high achievers among sequences whose limit is zero, and represents a minor extension of results appearing in [5], [6], and [10, Chapter 2].

Theorem 1.1. *Let $(x_n) = x_1, x_2, x_3, \ldots$ be a sequence of real numbers with $x_n \to 0$. Suppose that for each $k \geq 1$,*

$$|x_k| \leq \sum_{n=k+1}^{\infty} |x_n|. \tag{1}$$

Then (x_n) is a high achiever. Moreover, if $|x_k| \geq |x_{k+1}|$ for each $k \geq 1$ then (x_n) is a high achiever if and only if (1) holds.

Note that if one drops the requirement $|x_k| \geq |x_{k+1}|$ for each $k \geq 1$ then it is easy to find high achievers that violate (1): any nontrivial rearrangement of $(\frac{1}{2^n})$ suffices.

Before getting to the proof of Theorem 1.1, we give a lemma that will be used repeatedly in the sequel to handle sequences with negative terms.

Lemma 1.2. *Let (x_n) be a sequence of real numbers, and suppose that the sum of the negative terms of (x_n) converges to $s_N \leq 0$. Then $-s_N + AS(x_n) = AS(|x_n|)$.*

Proof. Partition $\mathbb{Z}^+$ into the disjoint subsets $I_P = \{j \mid x_j > 0\}$ and $I_N = \{j \mid x_j < 0\}$. Since I_P and I_N partition $\mathbb{Z}^+$ (recall our convention that the terms of all sequences are nonzero), we have $AS(x_n) = AS(x_i \mid i \in I_P) + AS(x_i \mid i \in I_N)$, where $+$ denotes the arithmetic sum. Note that in this equation, we use the fact that our hypothesis on the negative terms ensures that a subsequence with convergent sum must in fact have absolutely convergent sum. Taking absolute values then yields

$$AS(|x_n|) = AS(x_i \mid i \in I_P) - AS(x_i \mid i \in I_N). \tag{2}$$

Let $r \in AS(x_n)$, so that there is $K \subseteq \mathbb{Z}^+$ such that $r = \sum_{k \in K} x_k$. Then $K = K_P \cup K_N$ for some $K_P \subseteq I_P$ and $K_N \subseteq I_N$. Thus

$$r - s_N = \left(\sum_{i \in K_P} x_i + \sum_{i \in K_N} x_i \right) - \sum_{i \in I_N} x_i = \sum_{i \in K_P} x_i - \sum_{i \in I_N \setminus K_N} x_i$$

and by (2) this last expression is an element of $AS(|x_n|)$. We've therefore shown $AS(x_n) - s_N \subseteq AS(|x_n|)$.

To show the reverse inclusion, suppose that $r \in AS(|x_n|)$. By (2), there must be subsets $J_P \subseteq I_P$ and $J_N \subseteq I_N$ such that

$$r = \sum_{i \in J_P} x_i - \sum_{i \in J_N} x_i.$$

Adding and subtracting $\sum_{i \in I_N} x_i$ to the right-hand side gives

$$r = \left(\sum_{i \in J_P} x_i + \sum_{i \in I_N \setminus J_N} x_i \right) - s_N,$$

and thus $r \in AS(x_n) - s_N$. □

Proof of Theorem 1.1. Let I_N be as in the proof of Lemma 1.2, and assume first that $\sum_{i \in I_N} x_i$ converges. By Lemma 1.2 it is enough in this case to show that $(|x_n|)$ is a high achiever. We may thus assume that all terms of (x_n) are positive.

Let s denote the sum of the x_n. Clearly it is enough to show that $r \in AS(x_n)$ for $0 < r < s$.

We define indices $i_1, i_2, i_3, \ldots$ using a greedy algorithm. Let i_1 be the smallest index satisfying $x_{i_1} \leq r$. Inductively, if $i_1, i_2, \ldots, i_m$ are already chosen, we take i_{m+1} to be the smallest index such that $i_{m+1} > i_m$ and

$$x_{i_{m+1}} + \sum_{j=1}^{m} x_{i_j} \leq r,$$

provided that at least one such index exists.

If this process terminates, then there must be some m such that $i_1, \ldots, i_m$ are defined but for each $n > i_m$ we have $x_n + \sum_{j=1}^{m} x_{i_j} > r$. By construction $\sum_{j=1}^{m} x_{i_j} \leq r$, and by hypothesis $\lim_{n\to\infty} x_n = 0$. It follows that $\sum_{j=1}^{m} x_{i_j} = r$, whence $r \in AS(x_n)$.

Suppose now that the process of constructing the i_j does not terminate, and suppose further that the sequence $i_1, i_2, i_3, \ldots$ omits a finite number of positive integers. Since $r < s$ the sequence must omit at least one positive integer. Let k be the largest such integer. Consider the sum t of the x_{i_j} with $i_j < k$ (let $t = 0$ if there are no $i_j < k$). We then have $x_k + t > r$ and $t + \sum_{h=1}^{\infty} x_{k+h} \leq r$. It follows that $x_k > \sum_{h=1}^{\infty} x_{k+h}$, which contradicts (1).

Therefore if the process of constructing the i_j does not terminate, then the sequence $i_1, i_2, i_3, \ldots$ omits an infinite number of positive integers. Let $\{k_1, k_2, k_3, \ldots\}$ be such a sequence. This means that for each k_l,

$$x_{k_l} + \sum_{i_j < k_l} x_{i_j} > r \geq \sum_{i_j < k_l} x_{i_j}. \tag{3}$$

By hypothesis $\lim_{l\to\infty} x_{k_l} = 0$, and so taking the limit as $l \to \infty$ in (3) gives $r = \sum_{j=1}^{\infty} x_{i_j}$. Thus $r \in AS(x_n)$. This proves the theorem in the case that $\sum_{i\in I_N} x_i$ converges.

If $\sum_{i\in I_N} x_i$ diverges, then the positive-term sequence $(-x_i \mid i \in I_N)$ satisfies (1) for each k. Thus $AS(-x_i \mid i \in I_N) = [0, \infty)$. Letting I_P be the set of indices of the positive terms of (x_n), we now have

$$AS(x_n) = \begin{cases} (-\infty, c] & \text{if } \sum_{i\in I_p} x_i \text{ converges to } c \\ (-\infty, \infty) & \text{if } \sum_{i\in I_P} x_i \text{ diverges} \end{cases} \tag{4}$$

In either case, $AS(x_n)$ is a high achiever.

We now prove the second assertion of the theorem. Suppose that $|x_k| \geq |x_{k+1}|$ for each $k \geq 1$, and also that (1) does not hold, i.e., there exists an index k with

$$|x_k| > \sum_{n=k+1}^{\infty} |x_n|.$$

This implies that the terms of (x_n) form an absolutely convergent series, so by Lemma 1.2 we may assume without loss of generality that the terms of (x_n) are positive. Clearly both $b = x_k$ and $a = \sum_{n=k+1}^{\infty} x_n$ are in $AS(x_n)$. We claim that $AS(x_n) \cap (a, b)$ is empty, which shows that (x_n) is not a high achiever. Let $I \subseteq \mathbb{Z}^+$. If $i \in I$ for some $i \leq k$, then since $x_i \geq x_k$, we have $\sum_{i\in I} x_i \geq x_k = b$. On the other hand, if I omits every $j \leq k$, then $\sum_{i\in I} x_i \leq \sum_{i=k+1}^{\infty} x_i = a$. $\square$

We now reap some of the fruits of Theorem 1.1; see also [10, Chapter 2].

Corollary 1.3. $AS\left(\frac{1}{n}\right) = [0, \infty)$

Corollary 1.3 follows immediately from the fact that the harmonic series diverges, and implies that every real number in $[0, 1]$ can be expressed as a (possibly infinite) Egyptian fraction. Indeed, such an Egyptian fraction can even be taken with all denominators prime. This follows from the result of Euler that $\sum_{n=1}^{\infty} \frac{1}{p_n}$ diverges [9, p. 59], where $p_1, p_2, \ldots$ is an enumeration of the primes, implying that $AS(1/p_n) = [0, \infty)$.

We also have an analogue of Riemann's rearrangement theorem. Note that in our setting we allow only omissions of terms, not rearrangements.

Corollary 1.4. *Let (x_n) be a sequence whose terms form a conditionally convergent series. Then $AS(x_n) = \mathbb{R}$.*

Proof. Let I_P and I_N be, respectively, the set of indices of the positive and negative terms of (x_n). Conditional convergence implies $\sum_{i \in I_P} x_i = \infty$ and $\sum_{i \in I_N} x_i = -\infty$. Theorem 1.1 then shows that $AS(y_i \mid i \in I_P) = [0, \infty)$ and $AS(y_i \mid i \in I_N) = (-\infty, 0]$. The corollary follows immediately. □

Our final corollary gives us a practical method for showing that many sequences whose terms form absolutely convergent series are high achievers.

Corollary 1.5. *Let x_n be a sequence with $\lim_{n\to\infty} x_n = 0$, and suppose $|x_{n+1}| \geq \frac{1}{2}|x_n|$ for all n. Then (x_n) is a high achiever.*

Proof. By iterating our hypothesis, we have $|x_{k+i}| \geq \frac{1}{2^i}|x_k|$ for every k and i. Thus for each k

$$\sum_{i=1}^{\infty} |x_{k+i}| \geq \sum_{i=1}^{\infty} \frac{1}{2^i} |x_k| = |x_k| \sum_{i=1}^{\infty} \frac{1}{2^i} = |x_k| \, .$$

It follows from Theorem 1.1 that (x_n) is a high achiever. □

Corollary 1.5 may be applied to the sequence of Fibonacci reciprocals $(1/F_n)$. When summed, they yield a series converging to $\beta \approx 3.36$, a number of considerable mystery whose irrationality was proven only in 1989 [1]. Since $F_{n+1} = F_n + F_{n-1} \leq 2F_n$, Corollary 1.5 shows that $AS(\frac{1}{F_n}) = [0, \beta]$.

2 Cantor Sets.

As noted in the introduction, $AS(2/3^n)$ is the Cantor middle third set. We wish to understand what kinds of sequences have Cantor sets as their achievement sets, and in this section we give a sufficient condition that is similar to the one in Theorem 1.1 (see Theorem 2.1). Recall that a generalized Cantor set, which we refer to simply as a Cantor set, is a compact, perfect, totally disconnected subset of the real numbers. Any Cantor set can be constructed in a manner similar to the Cantor middle third set: begin with a compact interval and remove an open subinterval in such a way that two closed intervals of positive length remain. These two intervals are called the intervals of stage one. Perform a similar operation on each of the intervals of stage one, leaving four intervals of stage two. Continue in this way, producing 2^k disjoint intervals at stage k. If we let C_k be the union of the intervals of stage k, then $C = \bigcap_{k=0}^{\infty} C_k$ is a Cantor set.

For our purposes, we are interested primarily in *central Cantor sets*, namely those that can be formed by following the recipe of the previous paragraph, but all open subintervals removed at any given stage have the same length and must be centered. The Cantor middle-third set is an example.

Theorem 2.1. *Let (x_n) be a real sequence, and suppose that for each $k \geq 1$,*

$$|x_k| > \sum_{i=k+1}^{\infty} |x_i|. \tag{5}$$

Then $AS(x_n)$ is a central Cantor set.

The removed intervals of stage k all have length $|x_k| - \sum_{i=k+1}^{\infty} |x_i|$. It follows that every central Cantor set with 0 as its left endpoint is the achievement set of some sequence (see Section 4). Also, under the hypotheses of Theorem 2.1, the measure of $AS(x_n)$ is

$$\lim_{k\to\infty} 2^k \sum_{i=k+1}^{\infty} |x_i|.$$

For more on the interesting question of how the measure and Hausdorff dimension of $AS(x_n)$ relate to (x_n), see [7, 8].

In order to prove Theorem 2.1, it is certainly necessary to establish that $AS(x_n)$ is closed. This result has interest in its own right, and is originally due to Hans Hornich [5]. We give its proof as a separate theorem, following which we prove Theorem 2.1.

Theorem 2.2 (Hornich)**.** *Let (x_n) be a positive-term sequence, and suppose $\sum_{n=1}^{\infty} x_n$ converges. Then $AS(x_n)$ is closed.*

Proof. Let (s_j) be a sequence of elements of $AS(x_n)$ whose limit is s. Let $I_j \subseteq \mathbb{Z}^+$ satisfy $s_j = \sum_{i\in I_j} x_i$.

Suppose there are no positive integers n that belong to I_j for infinitely many j. Then fixing an $m > n_k$, we have that for all j sufficiently large,

$$s_j \leq \sum_{n=m+1}^{\infty} x_n.$$

Letting $m \to \infty$ and using the convergence of $\sum_{n=1}^{\infty} x_n$, we have $s = 0 \in AS(x_n)$.

Suppose now that there is a positive integer belonging to I_j for infinitely many j, and let n_1 be the smallest one. Thus there is an infinite set J_1 such that for all $j \in J_1$, n_1 is the smallest element of I_j. Now suppose that $n_1, \ldots, n_k$ have been chosen, and there is an infinite set J_k such that for all $j \in J_k$, the first k elements of I_j are $n_1, \ldots, n_k$.

If there are no $n > n_k$ that belong to I_j for infinitely many $j \in J_k$, then fixing an $m > n_k$ we have that for all $j \in J_k$ sufficiently large,

$$s_j - (x_{n_1} + \cdots + x_{n_k}) \leq \sum_{i=m+1}^{\infty} x_i. \tag{6}$$

Letting $m \to \infty$, we have $s = x_{n_1} + \cdots + x_{n_k} \in AS(x_n)$. If there is an $n > n_k$ that belongs to I_j for infinitely many j, then let n_{k+1} be the smallest one. Then there is an infinite set J_{k+1} such that for all $j \in J_k$, the first $k+1$ elements of I_j are $n_1, \ldots, n_{k+1}$.

This process either terminates or results in an infinite sequence $n_1, n_2, \ldots$. In the former case, from (6) we have $s \in AS(x_n)$. In the latter case, we can choose for each k some s_{j_k} with

$$s_{j_k} - (x_{n_1} + \cdots + x_{n_k}) \leq \sum_{i=n_k+1}^{\infty} x_i. \tag{7}$$

Since the original sequence (s_j) approaches s, the subsequence (s_{j_k}) must too. We thus obtain $s = \sum_{i=1}^{\infty} x_{n_i}$ by taking $k \to \infty$ in (6). Therefore $s \in AS(x_n)$, as desired. □

Proof of Theorem 2.1. An immediate consequence of (5) is that the series $\sum_{n=1}^{\infty} x_n$ is absolutely convergent. In particular, the sum of the negative terms of (x_n) must converge. Thus by Lemma 1.2, we may assume that each x_n is positive, since a translate of a central Cantor set is again a central Cantor set.

For $k \geq 1$, let $t_k = \sum_{i=k}^{\infty} x_i$. For $k \geq 0$, define C_k to be the set

$$AS(x_1, \ldots, x_k) + [0, t_{k+1}]. \tag{8}$$

Since $AS(x_1, \ldots, x_k)$ is a finite set, C_k is a union of closed intervals. We also clearly have $C_k \supseteq C_{k+1}$ for each $k \geq 0$.

We claim that $\bigcap_{k=0}^{\infty} C_k = AS(x_n)$. If $s \in AS(x_n)$, then it follows immediately that $s \in \bigcap_{k=0}^{\infty} C_k$. Suppose that $s \in C_k$. Then $s - q_k \in [0, t_{k+1}]$ for some $q_k \in AS(x_1, \ldots, x_k) \subseteq AS(x_n)$. In particular, $|s - q_k| \leq t_{k+1}$. Because $\sum_{n=1}^{\infty} x_n$ converges, $t_{k+1} \to 0$ as $k \to \infty$. So if we take $s \in \bigcap_{k=0}^{\infty} C_k$, then there exists an infinite sequence $q_1, q_2, \ldots$ of elements in $AS(x_n)$ such that $\lim_{k\to\infty} |s - q_k| = 0$. Because $AS(x_n)$ is closed by Theorem2.2, we have $s \in AS(x_n)$.

To complete the proof, we need only show that each C_k is central, i.e., that each consists of 2^k disjoint intervals and that C_{k+1} is formed from C_k by deleting a central open interval from each interval of C_k. Note that

$$C_1 = [0, t_2] \cup [x_1, x_1 + t_2] = [0, t_2] \cup [x_1, t_1] \subseteq [0, t_1] = C_0.$$

By (5), $x_1 > t_2$, so the intervals of C_1 are disjoint. Moreover, $t_2 = t_1 - x_1$, so the removed subinterval (t_2, x_1) is a central interval of $[0, t_1] = C_0$.

Now suppose inductively that C_k is central, which implies that it is a union of 2^k disjoint intervals. By definition, each interval of C_k is a translate of $[0, t_{k+1}]$. Thus C_{k+1} consists of disjoint pairs of intervals that are translates of $[0, t_{k+2}] \cup [x_{k+1}, t_{k+1}]$. By (5) we have $x_{k+1} > t_{k+2}$, so each interval in a given pair is disjoint and the removed subinterval has the same length. Because $t_{k+2} = t_{k+1} - x_{k+1}$, the removed subinterval (t_{k+2}, x_{k+1}) is central. Hence C_{k+1} is central. Thus by induction all the C_k are central. $\square$

Using Theorem 2.1 as well as results from Section 1, we can generalize Theorem 2.2.

Corollary 2.3. *Suppose* $\lim_{n\to\infty} x_n = 0$. *Then* $AS(x_n)$ *is closed.*

Proof. Let $s_N \geq -\infty$ denote the sum of the negative terms of (x_n). If s_N is infinite, then as in the proof of Theorem 1.1 (see (4)) $AS(x_n)$ is closed. If s_N is finite then by Lemma 1.2 we can assume that (x_n) is positive term, because a translate of a closed set is again closed. If $\sum_{n=1}^{\infty} x_n$ converges, then $AS(x_n)$ is closed by Theorem 2.2. If $\sum_{n=1}^{\infty} x_n$ diverges then $AS(x_n) = [0, \infty)$ by Theorem 1.1. $\square$

It is not true that all achievement sets are closed. For instance, suppose that $x_n = 1 + 1/n$ for all $n \geq 1$. Then $AS(x_n)$ does not contain its limit point 1. In this example, $AS(x_n)$ is countable, so it is natural to ask if all uncountable achievable sets are closed; the following example of Velleman [11], shows that the answer is no.

Consider the two sequences given by

$$x_n = \frac{2}{3^n} \qquad \text{and} \qquad y_n = 2 - \frac{1}{2 \cdot 3^{n-1}}.$$

Make a new sequence z_n by interleaving these two, so that the first few terms of z_n are $2/3, 3/2, 2/9, 11/6, 2/27, 35/18$. Note that $AS(x_n)$ is the usual Cantor one-third set, and hence $AS(z_n)$ is uncountable. Moreover, 2 is an accumulation point of $AS(y_n)$, and thus also of $AS(z_n)$. We now show that $2 \notin AS(z_n)$. Suppose a subsequence of z_n sums to 2, and note that it can contain at most one term of (y_n), since $y_n > 1$ for all n. Moreover, this subsequence must contain at least one term of (y_n), since summing all the x_n yields 1. Therefore we have a subsequence of (x_n) whose terms sum to $\frac{1}{2 \cdot 3^{k-1}}$ for some k. However, $\frac{1}{2 \cdot 3^{k-1}}$ is halfway between $\frac{1}{3^k}$ and $\frac{2}{3^k}$, and so is not contained in the Cantor one-third set. This is a contradiction, proving that $2 \notin AS(z_n)$.

3 The Two Kinds of Achievement Sets.

Thus far we have seen that many achievement sets are intervals, and thus connected (Section 1), and many are Cantor sets and thus totally disconnected (Section 2). Examples of achievement sets that are unions of disjoint intervals also abound: for instance, if $x_1 = 2$ and $x_n = (\frac{1}{2})^{n-1}$ for $n \geq 2$, then

$AS(x_n) = [0,1] \cup [2,3]$. This raises the question of whether there are achievement sets that contain an interval but are not unions of intervals. In fact, there are such achievement sets, and we give one that comes from [11].

The idea is to construct an achievement set that consists of all numbers representable by a certain kind of base a expansion, where a is a suitably chosen real number less than 1. The allowable multiples of the powers of a come from a set having additive properties that ensure no intervals are contained in the extremities of the achievement set, but an interval is contained in the middle.

Define $x_1 = \frac{3}{5}$, $x_2 = \frac{2}{5}$, $x_3 = \frac{2}{5}$, $x_4 = \frac{2}{5}$, and for $n > 4$ put $x_n = a \cdot x_{n-4}$, where a is chosen so that

$$\frac{1}{5} \leq \sum_{n=1}^{\infty} a^n < \frac{2}{9} \tag{9}$$

For instance, $a = 19/109$ will do. Put $b = \sum_{n=1}^{\infty} a^n = \frac{a}{1-a}$, and note that $AS(x_n) \subseteq \left[0, \frac{9}{5}(1+b)\right]$.

By (9) we have $\frac{9}{5}\sum_{n=1}^{\infty} a^n < \frac{2}{5}$, whence $AS(x_n)$ omits the interval $\left(\frac{9}{5}b, \frac{2}{5}\right)$. Similarly, $AS(x_n)$ omits the intervals $\left(\frac{9}{5}ba^i, \frac{2}{5}a^i\right)$ for all $i \geq 1$. Thus $0 \in AS(x_n)$ but $AS(x_n)$ omits an interval in all neighborhoods of 0. It follows that $AS(x_n)$ is not a finite union of intervals.

On the other hand, we claim $\left[\frac{2}{5}(1+b), \frac{7}{5}(1+b)\right] \subset AS(x_n)$. Consider the sequence y_n defined by $y_n = \frac{1}{5}a^i$ for $5i+1 \leq n \leq 5i+5$. Thus the first five terms of y_n are all $1/5$, the next five are all $a \cdot 1/5$, the next five are $a^2 \cdot 1/5$, and so on. By Theorem 1.1, $AS(y_n)$ must be an interval provided that for all i we have

$$\frac{1}{5}a^i \leq \frac{5}{5}\sum_{n=i+1}^{\infty} a^n.$$

This holds by (9), and therefore $AS(y_n) = [0, 1+b]$.

Now let $c \in \left[\frac{2}{5}(1+b), \frac{7}{5}(1+b)\right]$. Then c can be written as $\frac{2}{5}(1+b)$ plus an element of $AS(y_n)$. We thus have

$$\begin{aligned} c &= \frac{2}{5}\sum_{n=0}^{\infty} a^n + \left[\frac{k_0}{5} + \frac{k_1}{5}a + \frac{k_2}{5}a^2 + \cdots\right] \qquad 0 \leq k_n \leq 5 \\ &= \frac{2+k_0}{5} + \frac{2+k_1}{5}a + \frac{2+k_2}{5}a^2 + \cdots \qquad 0 \leq k_n \leq 5 \end{aligned}$$

All fractions of the form $(2+k)/5$, $0 \leq k \leq 5$ may be produced by summing subcollections of $\{3/5, 2/5, 2/5, 2/5\}$. Therefore $c \in AS(x_n)$, proving that $\left[\frac{2}{5}(1+b), \frac{7}{5}(1+b)\right] \subset AS(x_n)$.

We remark that by Theorem 2.1 the sequences $(\frac{3}{5}a^n)$ and $(\frac{2}{5}a^n)$ both have achievement sets that are central Cantor sets. Therefore $AS(x_n)$ is the arithmetic sum of four central Cantor sets, and we have shown that this arithmetic sum can contain an interval without being a disjoint union of intervals. In general, the question of when an arithmetic sum of Cantor sets contains an interval is difficult; see e.g. [3].

We can, however, salvage some kind of dichotomy among achievement sets, thanks to a result mainly from [11]. Recall that a set is nowhere dense if its closure has empty interior, and *meager* (or *of first category*) if it is a countable union of nowhere dense sets. By the Baire Category Theorem a meager subset of the reals has empty interior, and hence is totally disconnected.

Theorem 3.1. *Let (x_n) be a sequence of real numbers. Then $AS(x_n)$ is either a meagre set, and thus has empty interior, or the interior of $AS(x_n)$ is dense in $AS(x_n)$.*

Before embarking on the proof of Theorem 3.1, we give a proposition that effectively reduces the proof to the case where $x_n \to 0$. This proposition has some independent interest as well, and gives some justification for our emphasis thus far on sequences whose terms approach 0.

Proposition 3.2. *Let (x_n) be any real sequence. Then $AS(x_n)$ is either countable, an infinite interval, or a countable union of translates of $AS(x_{n_k})$, where $\lim_{n\to\infty} x_{n_k} = 0$.*

Proof. Let E be the set of accumulation points of (x_n). Suppose first that $E \cap (0, \epsilon) \neq \emptyset$ for every $\epsilon > 0$. Then there is a sequence $e_1, e_2, \ldots$ of elements of e with $e_n \to 0$ and $e_n < 1$ for all n. For each n, let k_n be a positive integer with $1/k_n > e_n > 1/(k_n + 2)$, whence there are infinitely many x_j with $1/k_n > x_j > 1/(k_n + 2)$. For each n, choose $k_n + 2$ such terms and form an infinite subsequence by concatenation. This subsequence approaches 0 but its sum diverges, and thus it has achievement set $[0, \infty)$ by Theorem 1.1. It follows that $AS(x_n)$ is an infinite interval. In the case where $E \cap (-\epsilon, 0) \neq \emptyset$ for every $\epsilon > 0$ a similar argument applies.

Now suppose that there is some $\epsilon > 0$ such that $E \cap \{r \in \mathbb{R} \mid 0 < |r| < \epsilon\} = \emptyset$. Let (x_{n_k}) be the subsequence consisting of the terms of (x_n) with absolute value at least $\frac{\epsilon}{2}$. Let (x_{m_k}) be the complementary subsequence. Note that

$$AS(x_n) = AS(x_{n_k}) + AS(x_{m_k}).$$

The first summand on the right-hand side consists only of sums of finitely many terms, and thus is countable (see also Proposition 4.1). By our assumption about E, the only possible accumulation point of (x_n) in $(-\epsilon, \epsilon)$ is 0, so the sequence (x_{m_k}) is either finite or has a limit of 0. In the first case, $AS(x_n)$ is countable, while in the second it is a countable union of translates of $AS(x_{n_j})$. $\square$

Note that in the case that $AS(x_n)$ is an infinite interval, it is clearly a countable union of translates of $AS(\frac{1}{2^n})$. Thus Proposition 3.2 implies that $AS(x_n)$ is either countable or a countable union of translates of $AS(y_n)$, where (y_n) is some sequence whose terms approach zero. We are now ready to prove Theorem 3.1.

Proof of Theorem 3.1. We first remark that translates of a meager set are meager, and a countable union of meager sets is also meager. Moreover, it is easy to see that the same two statements hold if "meager" is replaced by "has dense interior." Thus by Proposition 3.2, it is enough to prove the theorem in the case that $x_n \to 0$.

Assume now that the theorem is true when $x_n \to 0$ and $x_n > 0$. If $x_n \to 0$ but x_n has both positive and negative terms, consider the sum of the negative terms. If this sum diverges then by Theorem 1.1, $AS(x_n)$ is an interval and thus has dense interior. If it converges, then by Lemma 1.2 we have that $AS(x_n)$ is a translate of $AS(|x_n|)$. By assumption we have that $AS(|x_n|)$ has either empty interior or dense interior, so the set $AS(x_n)$ must also fall into one of these two classes.

Therefore it suffices to prove the theorem under the hypotheses that $x_n \to 0$ and $x_n > 0$ for all n. In this case we first show that if 0 is in the closure of the interior of $AS(x_n)$, then $AS(x_n)$ has dense interior. Let $x \in AS(x_n)$, and fix $\epsilon > 0$. Since $x \in AS(x_n)$, there is a subsequence of (x_n) whose terms sum to x. Therefore we can find a finite sum $x_{n_1} + \cdots + x_{n_k}$ such that $x - \epsilon < x_{n_1} + \cdots + x_{n_k} \leq x$. Let $\delta = \min\{\epsilon, x_{n_1}, \ldots, x_{n_k}\}$. Since 0 is in the closure of the interior of $AS(x_n)$, we can find a and b so that $0 < a < b < \delta$ and $(a, b) \subseteq AS(x_n)$. Notice that every element of (a, b) can be written as the sum of a subsequence of (x_n), but the terms $x_{n_1}, \ldots, x_{n_k}$ will not be used in any of these sums, because they are too large. It follows that $x_{n_1}+\cdots+x_{n_k}+(a, b) \subseteq AS(x_n)$. But $x_{n_1} + \cdots + x_{n_k} + (a, b) \subseteq (x - \epsilon, x + \epsilon)$. Since ϵ was arbitrary, this shows that x is in the closure of the interior of $AS(x_n)$.

Now we show that if 0 is not in the closure of $AS(x_n)$, then $AS(x_n)$ is meager. In this case there is some $\epsilon > 0$ such that $[0, \epsilon)$ contains no elements of the interior of $AS(x_n)$. Therefore $AS(x_n)$ is not a high achiever, and it follows from Theorem 1.1 that $\sum_{n=1}^{\infty} x_n$ converges. Hence we can choose some N such that $\sum_{n=N}^{\infty} x_n < \epsilon$. Since $AS(x_N, x_{N+1}, \ldots) \subseteq AS(x_n) \cap [0, \epsilon)$, we have that $AS(x_N, x_{N+1}, \ldots)$ has empty interior, and moreover by Theorem 2.2 it is closed and thus is nowhere dense. Now

$$AS(x_n) = AS(x_1, \ldots, x_{N-1}) + AS(x_N, x_{N+1}, \ldots),$$

and since $AS(x_1, \ldots, x_{N-1})$ is finite, $AS(x_n)$ is a finite union of nowhere dense sets, and hence is meager. $\square$

If we restrict our attention to certain sequences, we can recover a dichotomy stronger than that given in Theorem 3.1. For instance, $AS(\frac{1}{c^n})$ is an interval if $1 < c \leq 2$ (Corollary 1.5), and a central Cantor set for $c > 2$ (Theorem 2.1). In fact, we can generalize this remark:

Proposition 3.3. *Let (x_n) be a real sequence, and suppose $\lim_{n\to\infty} \left|\frac{x_{n+1}}{x_n}\right|$ exists and equals L. Then $AS(x_n)$ is a finite union of closed intervals if $\frac{1}{2} < L < 1$ and a finite union of central Cantor sets if $0 \leq L < \frac{1}{2}$.*

Proof. Note that since $L < 1$, $\lim_{n\to\infty} x_n = 0$. If $\frac{1}{2} < L < 1$, then for some $n_0 > 0$ we have $\frac{|x_{n+1}|}{|x_n|} > \frac{1}{2}$ for all $n \geq n_0$. Hence by Corollary 1.5, we have that $AS(x_{n_0}, x_{n_0+1}, x_{n_0+1}, \ldots)$ is a closed interval. It then follows from the decomposition

$$AS(x_n) = AS(x_1, \ldots, x_{n_0-1}) + AS(x_{n_0}, x_{n_0+1}, x_{n_0+1}, \ldots)$$

that $AS(x_n)$ is the union of a finite number of translates of a closed interval.

If $0 \leq L < \frac{1}{2}$, then for some $n_0 > 0$, we have $\frac{|x_{n+1}|}{|x_n|} < \frac{1}{2}$ for all $n \geq n_0$. Thus for any $i \geq 1$ and any $n \geq n_0$, we have $|x_{n+i}| < \frac{1}{2^i}|x_n|$. Therefore

$$\sum_{i=1}^{\infty} |x_{n+i}| < \sum_{i=1}^{\infty} \frac{1}{2^i} |x_n| = |x_n| \sum_{i=1}^{\infty} \frac{1}{2^i} = |x_n| .$$

It now follows from Theorem 2.1 that $AS(x_{n_0}, x_{n_0+1}, x_{n_0+1}, \ldots)$ is a central Cantor set. Therefore $AS(x_n)$ is a finite union of central Cantor sets. □

We can extend Proposition 3.3 in a few ways. If $L > 1$, then no infinite subsequence of (x_n) can have a convergent sum, and thus $AS(x_n)$ is countable. In addition, if $L = 1$ and $\lim_{n\to\infty} x_n = 0$, then $AS(x_n)$ is a finite union of closed intervals by the same argument used in the case $\frac{1}{2} < L < 1$. However, if $L = \frac{1}{2}$ or if $L = 1$ and $\lim_{n\to\infty} x_n \neq 0$, many behaviors are possible.

To illustrate the variety of behaviors possible when $L = \frac{1}{2}$, consider the three sequences $\left(\frac{1}{2^n} - \frac{1}{3^n}\right)$, $\left(\frac{1}{2^n} + \frac{1}{3^n}\right)$, and $\left(\frac{1}{2^n} + \frac{1}{(-3)^n}\right)$. One can easily verify that the first satisfies, for each k, condition (1) of Theorem 1.1, namely $|x_k| \leq \sum_{n=k+1}^{\infty} |x_n|$. Thus its achievement set is a closed interval. Similarly, the second satisfies, for each k, condition (5) of Theorem 2.1, namely $|x_k| > \sum_{n=k+1}^{\infty} |x_n|$. Thus its achievement set is a Cantor set.

However, the third sequence has a mysterious achievement set. The sequence satisfies (1) for k odd and (5) for k even, and we give an intuitive description of how this alternation affects the achievement set. Recall the sets C_k as defined in (8) on p. 7. For a general sequence (x_n), each C_k consists of 2^k not necessarily disjoint intervals, while C_{k+1} is formed by splitting each interval of C_k into two not necessarily disjoint intervals, which we refer to here as "new intervals." If (1) holds for k then each pair of new intervals is overlapping, while if (5) holds for k then each pair of new intervals is disjoint. Each pair of new intervals is disjoint when k is even and overlapping when k is odd. Because C_k is the union of all new intervals at stage k, the gaps that are introduced when k is even may be covered by the overlap of the previous stage. See Figure 1 for an illustration.

This interaction appears to be complicated. In particular, it is not clear if any intervals survive all stages unpunctured (though it is easy to see that $AS(x_n)$ is not itself an interval).

Question 1. *Let* $(x_n) = \left(\frac{1}{2^n} + \frac{1}{(-3)^n}\right)$. *Does* $AS(x_n)$ *have empty interior? Give a topological description of* $AS(x_n)$.

Note that from the proof of Theorem 3.1, one sees that to answer Question 1, it is enough to determine whether 0 is in the closure of the interior of $AS(x_n)$.

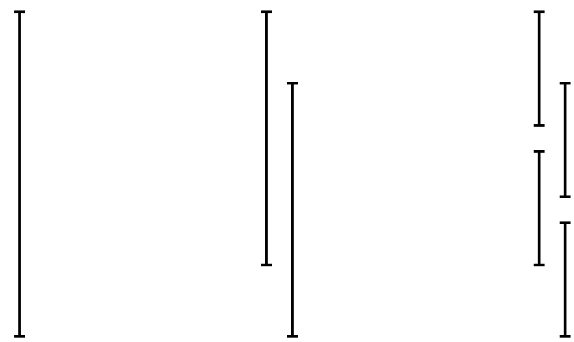

Figure 1: Intervals in three successive stage approximations to $\left(\frac{1}{2^n} + \frac{1}{(-3)^n}\right)$

In other words, does there exist $\epsilon > 0$ such that $AS(x_n) \cap [0, \epsilon]$ contains no elements of the interior of $AS(x_n)$?

In the same vein as Question 1, we pose the following:

Question 2. *Does there exist a sequence* (x_n) *such that* $\lim_{n\to\infty} |x_{n+1}/x_n|$ *exists and* $AS(x_n)$ *contains an interval but is not a union of intervals?*

4 Achievable Sets.

Call a subset of $\mathbb{R}$ *achievable* if it can be obtained as $AS(x_n)$ for some real sequence (x_n). In this section we give some examples of classes of achievable sets, and examine the achievability of some well-known subsets, such as $\mathbb{Q}$.

As a simple starting example, any closed interval with 0 as an endpoint is achievable: for $r \in \mathbb{R}$, we have $AS(\frac{r}{2^n}) = [0, r]$ (or $[r, 0]$ if $r < 0$). Now let $s \in \mathbb{R}$, and consider the sequence $(x_n) = s, \frac{r}{2}, \frac{r}{2^2}, \ldots$. Then $AS(x_n) = [0, r] \cup [s, s+r]$, with appropriate alterations made for $r < 0$. More generally if S is any achievable set and $r \in \mathbb{R}$, then both $\bigcup_{s\in S}[s, s+r]$ and $rS = \{rs : s \in S\}$ are achievable.

Consider now a central Cantor set C whose original interval has its left endpoint at 0. To specify such a set, one needs only the length L of the original interval and the length a_n of each of the central intervals removed at stage n. If (x_n) satisfies $|x_n| > \sum_{k=n+1}^{\infty} |x_k|$ for each n, Theorem 2.1 shows that $AS(x_n)$ is a central Cantor set and the length of each of the 2^n removed intervals at stage n is $|x_n| - \sum_{k=n+1}^{\infty} |x_k|$. Thus one constructs a sequence (x_n) with $AS(x_n) = C$ by taking $x_1 = \frac{1}{2}(L + a_1)$ and

$$x_n = \frac{L + 2^{n-1}a_n - \sum_{k=1}^{n-1} 2^{k-1}a_k}{2^n}$$

for each $n \geq 2$. It is straightforward to check that $x_1 + x_2 + \cdots = L$ and $x_n - x_{n+1} - x_{n+2} - \cdots = a_n$ for each n.

We now give some properties of achievable sets. Theorem 2.2 shows that any bounded achievable set must be closed, and Theorem 3.1 shows an achievable set must be meager or have dense interior. Any bounded achievable set must also be symmetric about its midpoint. Indeed, such a set S is the achievement set of (x_n), where $\sum_{n=1}^{\infty} x_n$ converges absolutely to r. if s is in $AS(x_n)$, then $s = \sum_{i \in I} x_i$ for some $I \subseteq \mathbb{Z}^+$. Letting $J = \mathbb{Z}^+ \backslash I$, we have $r - s = \sum_{j \in J} x_j$, which is in $AS(x_n)$, showing that $AS(x_n)$ is symmetric about $\frac{r}{2}$.

Our next two results furnish additional properties of achievable sets.

Proposition 4.1. *Let (x_n) be an infinite real sequence. Then $AS(x_n)$ is uncountable if and only if (x_n) has a subsequence converging to* 0.

Proof. Suppose first that (x_n) contains a subsequence converging to 0. Without loss of generality we may assume that $x_n \to 0$; we show that $AS(x_n)$ is uncountable.

If there is a k_0 such that whenever $k > k_0$ we have

$$|x_k| \leq \sum_{n=k+1}^{\infty} |x_n|, \tag{10}$$

then it follows from Theorem 1.1 that $AS(x_{n_k})$ contains an interval, and is thus uncountable. If there is no k_0 such that (10) is satisfied for $k \geq k_0$, then there must be a sequence $k_1, k_2, \ldots$ such that for each k_j,

$$|x_{k_j}| > \sum_{n=k_j+1}^{\infty} |x_n| \geq \sum_{i=j+1}^{\infty} |x_{k_i}|.$$

By Theorem 2.1 $AS(x_{k_j})$ is a central Cantor set, and thus uncountable.

Now suppose that (x_n) contains no subsequence converging to 0. Then no infinite sum of terms can converge, so all elements of $AS(x_n)$ are finite sums of terms. Hence $AS(x_n)$ is countable. □

Proposition 4.2. *If $AS(x_n)$ is uncountable, then it is without isolated points.*

Proof. Note that by Proposition 4.1 there is a subsequence (x_{n_j}) whose limit is 0. let $s = \sum_{i \in I} x_i \in AS(x_n)$. If I is finite, let k be its greatest element and let l be minimal such that $x_{n_l} > k$. Then $s + x_{n_l}, s + x_{n_{l+1}}, s + x_{n_{l+2}}, \ldots$ is an infinite sequence of elements of $AS(x_n)$ converging to s. If I is infinite, the partial sums of $\sum_{i \in I} x_i$ form an infinite sequence of elements of $AS(x_n)$ converging to s. Therefore $AS(x_n)$ has no isolated points. □

With these properties of achievable sets now established, we can examine the achievability of certain well-known sets.

Corollary 4.3. *If $S \subset \mathbb{R}$ is a countable set of nonnegative numbers having* 0 *as an accumulation point, then S is not achievable. In particular, the set of nonnegative rational numbers $\mathbb{Q}^+$ is not achievable.*

Proof. Suppose $AS(x_n) = S$. Clearly (x_n) can have no negative terms. Thus if $x_n > \epsilon$ for all n and some $\epsilon > 0$, then $AS(x_n) \cap (0, \epsilon)$ is empty, which contradicts the fact that 0 is an accumulation point of S. Hence (x_n) must have a subsequence converging to 0. By Proposition 4.1, $AS(x_n)$ is then uncountable, and we have a contradiction. □

It is interesting to note that if we adjoin a single negative number to $\mathbb{Q}^+$ the resulting set is achievable. For instance, let $q_1, q_2, \ldots$ be an enumeration of $\mathbb{Q}^+$. Set $x_1 = -1$ and for $n \geq 2$, let $x_n = 1 + q_{n-1}$. Clearly $\mathbb{Q}^+ \subseteq AS(x_n)$ and $-1 \in AS(x_n)$. But the terms of any finite subsequence of (x_n) sum to -1 or a nonnegative rational number, while the terms of any infinite subsequence form a divergent series. Hence $AS(x_n) = \{-1\} \cup \mathbb{Q}^+$. Using similar reasoning one can show that if G is any countably infinite additive subgroup of $\mathbb{R}$ and $g \in G^+$, then $\{-g\} \cup G^+$ is achievable.

The full set $\mathbb{Q}$ of rationals is also achievable. Let (x_n) be an enumeration of the rationals with absolute value at least 1. Since (x_n) has no subsequences with limit 0, no infinite sum of terms can converge. Finite sums of terms are rational numbers, so $AS(x_n) \subseteq \mathbb{Q}$. On the other hand, clearly $AS(x_n)$ contains all rationals of absolute value at least one. If $q \in \mathbb{Q} \cap (-1, 1)$, then we have $2 + (q - 2) \in AS(x_n)$. Thus $AS(x_n) = \mathbb{Q}$. A similar result can be shown for any countably infinite subgroup of $\mathbb{R}$.

Let us now consider $\mathbb{I} = \{r \in \mathbb{R} \mid r \text{ irrational}\} \cup \{0\}$. The interior of $\mathbb{I}$, being empty, cannot be dense in $\mathbb{I}$. Hence if $\mathbb{I}$ is achievable, then by Theorem 3.1 it must be meager. To see that this cannot be the case, note that the rationals are meager, and since unions of meager sets are again meager, we have that $\mathbb{R}$ is meager. But complete metric spaces cannot be meager by the Baire category theorem. A similar argument applies to the positive irrationals $\mathbb{I}^+$.

Acknowledgments. I am very grateful to Dan Velleman for his interest in this project, and his contributions of Theorem 3.1 and the examples at the end of Section 2 and the beginning of Section 3. These have enriched the paper substantially. I would also like to thank the anonymous referees for valuable suggestions.

References

[1] R. André-Jeannin, Irrationalité de la somme des inverses de certaines suites récurrents, *C. R. Acad. Sci. Paris Sér. I Math.* **308** (1989) 539–541.

[2] J. L. Brown, Jr., Note on complete sequences of integers, *Amer. Math. Monthly* **68** (1961) 557–560.

[3] C. Cabrelli, K. Hare, and U. Molter, Sums of Cantor sets yielding an interval, *J. Aust. Math. Soc.* **73** (2002) 405–418.

[4] V. E. Hoggatt and C. King, Problem E 1424, *Amer. Math. Monthly* **67** (1960) 593, Solution by J. Silver, **68** (1961), 179–180.

[5] H. Hornich, Über beliebige Teilsummen absolut konvergenter Reihen, *Monatsh. Math. Phys.* **49** (1941) 316–320.

[6] S. Kakeya, On the partial sums of an infinite series, *Sci. Rep. Tôhoku Imperial Univ.* **3** (1914) 159–163.

[7] M. Morán, Fractal series, *Mathematika* **36** (1989) 334–348 (1990).

[8] ———, Dimension functions for fractal sets associated to series, *Proc. Amer. Math. Soc.* **120** (1994) 749–754.

[9] T. Nagell, *Introduction to Number Theory*, John Wiley & Sons Inc., New York, 1951.

[10] P. Ribenboim, *My Numbers, My Friends*, Springer-Verlag, 2000.

[11] D. J. Velleman, personal communication, 2006.

Rafe Jones received his B. A. from Amherst College in 1998 and his Ph. D. from Brown University in 2005 under the direction of Joseph Silverman. He currently teaches at the College of the Holy Cross. While his mathematical interests lie mainly in arithmetic questions related to the iteration of rational functions, he is never averse to a good side project. When not doing mathematics, he enjoys running, cycling, speaking French, and the occasional computer-based distraction. These pastimes are not new; he hopes that none of his Amherst College professors noticed his world ranking (at the time) in Minesweeper.
Department of Mathematics and Computer Science, College of the Holy Cross, Worcester, MA, 01610
rjones@holycross.edu