1. Here is a truth table involving the two propositions:

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<th>Q ∨ R</th>
<th>P ⇒ (Q ∨ R)</th>
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Comparing the fifth column and the last column, we see that they are the same, and hence the two propositions are logically equivalent.

2. There exists a positive integer $n$ such that for all positive integers $k$, either $k$ is not prime or $k^2 > n$.

3. We argue by induction on $n$.

When $n = 1$, the claim is that $\frac{1}{1 \cdot 5} = \frac{1}{4(1) + 1}$, which is clearly true.

Now suppose that $k \geq 1$ is given, and that the result holds for $k$. Then our induction hypothesis says that

$$\frac{1}{1 \cdot 5} + \frac{1}{5 \cdot 9} + \cdots + \frac{1}{(4k-3)(4k+1)} = \frac{k}{4k+1}.$$

Now note that $\frac{1}{(4(k + 1) - 3)(4(k + 1) + 4)} = \frac{1}{(4k + 1)(4k + 5)}$, and when we add this to both sides of the equation above, we find

$$\frac{1}{1 \cdot 5} + \frac{1}{5 \cdot 9} + \cdots + \frac{1}{(4k-3)(4k+1)} + \frac{1}{(4k+1)(4k+5)} = \frac{k}{4k+1} + \frac{1}{4k+5} = \frac{k(4k+5)+1}{(4k+1)(4k+5)} = \frac{(4k+1)(k+1)}{(4k+1)(4k+5)} = \frac{k+1}{4k+5}.$$

This chain of equations establishes the result for $k + 1$, and completes the proof.

4. (a) This statement is false. Here is a counterexample: let $A = \{1\}$ and $B = \{2\}$. Then we have $\mathcal{P}(A \cup B) = \emptyset, \{1\}, \{2\}, \{1, 2\}$ and $\mathcal{P}(A) \cup \mathcal{P}(B) = \emptyset, \{1\}, \{2\}$. Note that $\{1, 2\} \in \mathcal{P}(A \cup B)$ but $\{1, 2\} \notin \mathcal{P}(A) \cup \mathcal{P}(B)$. 


5. To prove that a number is not rational, it’s a good idea to use an indirect proof. Let’s use a proof by contradiction. Suppose to the contrary that \( \log_3(5) \) is rational. Then there exist integers \( p \) and \( q \) such that \( \log_3(5) = p/q \). Note that since \( \log_3(5) \) is positive (in fact, since \( 5 > 3 \), it is at least 1), we in fact have that both \( p \) and \( q \) are positive. By the definition of logarithm, \( \log_3(5) = p/q \) means that \( 3^{p/q} = 5 \). Raising both sides to the \( q \) power then gives \( 3^p = 5^q \). But \( 3^p \) is divisible by 3 (because \( p \) is a positive integer), and \( 5^q \) is not. This contradiction proves the theorem.

6. (a) \( R \) is not symmetric, since \((2, 1) \in R \) but \((1, 2) \notin R \). \( R \) is not anti-symmetric, since \((1, 3) \in R \) and \((3, 1) \in R \) but \(1 \neq 3 \).

(b) To be systematic about this, we need to first consider all \((x, y) \in R \) with \( x = 1 \). This gives just one pair, namely \((1, 3) \), so \( y = 3 \). Now let’s find all \((y, z) \) with \( y = 3 \), namely \((3, 1) \) and \((3, 3) \). Transitive requires that since \( R \) contains the two pairs \((1, 3) \) and \((3, 1) \), then \( S \) must contain \((1, 1) \). Also, since \( R \) contains \((1, 3) \) and \((3, 3) \), \( S \) must contain \((1, 3) \); but \( R \) already has \((1, 3) \), so we don’t need to add anything.

So we must add \((1, 1) \) to \( R \) to get the relation \{\((1, 1), (2, 1), (1, 3), (3, 1), (3, 3), (4, 1)\)\}. Considering all \((x, y) \in R \) with \( x = 1 \) as before, we see that we do not need to add any elements to the new relation. So consider all \((x, y) \in R \) with \( x = 2 \). The only one that needs attention is the pair \((2, 1) \) and \((1, 3) \); we must add \((2, 3) \), which gives a new relation

\[ \{ (1, 1), (2, 1), (2, 3), (1, 3), (3, 1), (3, 3), (4, 1) \} \]

Now if we considering all \((x, y) \in R \) with \( x = 1 \) or \( x = 2 \), we need to add no new elements. The same goes for \( x = 3 \). To account for the pairs with \( x = 4 \), we need to add \((4, 3) \). Thus we take

\[ S = \{ (1, 1), (2, 1), (2, 3), (1, 3), (3, 1), (3, 3), (4, 1), (4, 3) \} \]

7. (a) Reflexive: Yes, since \((A - A) = \emptyset \), so \((A - A) \cup (A - A) = \emptyset \).

(b) Symmetric: Yes, since \( S \cup T = T \cup S \) for any sets \( S \) and \( T \), and so \( A \cap B \) implies \((A - B) \cup (B - A) = \emptyset \), which implies \((B - A) \cup (A - B) = \emptyset \), which implies \( B \cap A \).

(c) Anti-symmetric: Yes. Suppose that \( A \cap B \) and \( B \cap A \). Then \((A - B) \cup (B - A) = \emptyset \), which implies that both \((A - B) \) and \((B - A) \) are empty, for if either contained an element, then \((A - B) \cup (B - A) \neq \emptyset \). Now \((A - B) \) being empty implies that there does not exist \( x \) with \( x \in A \) and \( x \notin B \). Hence for all \( x \), either \( x \notin A \) or \( x \in B \). Therefore if \( x \in A \), then we must have \( x \in B \). This proves that \( A \subseteq B \). The same argument, with the roles of \( A \) and \( B \) reversed, shows that if \((B - A) \) is empty, then \( B \subseteq A \). We’ve now shown that \((A - B) \cup (B - A) = \emptyset \) implies \( A = B \). This proves that \( R \) is anti-symmetric.

(d) Transitive: Yes. Suppose that \( A \cap B \) and \( B \cap C \). In part (c), we showed that this implies \( A = B \) and \( B = C \). Hence \( A = C \), and so by part (a), \( A \cap C \). Hence \( R \) is transitive.