Carleton College, Winter 2014
Math 232, Practice Midterm 1
Prof. Jones

This practice exam is a little bit longer than the midterm will be. So give yourself 75 minutes to complete these problems. Solutions will be posted later this week. This is similar in format to the actual exam: the only difference is the actual exam will have a short answer section that includes several problems like the T/F review problems.

1. Find all solutions of the following system of linear equations.

$$\begin{array}{rrrrl} x_1 & +2x_2 & & +x_4 & = 0 \\ -3x_1 & & +x_3 & & = 1 \\ -x_1 & +4x_2 & +x_3 & +2x_4 & = 1 \end{array}$$

2. Let A be an $n \times n$ matrix of rank n, and suppose that $\mathbf{b}$ is a vector in $\mathbb{R}^n$. Must there be $\mathbf{v} \in \mathbb{R}^n$ with $A\mathbf{v} = \mathbf{b}$? Prove or give a counter-example.

3. For each of the following subsets W of a vector space V, determine if W is a subspace of V. In each case either prove that W is a subspace or give a concrete reason why it is not a subspace. [Hint: if you think one of these is a subspace, try to write it as the kernel or the image of a matrix.]

   (a) $V = \mathbb{R}^4$, and $W = \{(x_1, x_2, x_3, x_4) : x_1 = x_3 - x_2, \text{ and } x_4 = 0\}$

   (b) $V = \mathbb{R}^4$, and $W = \{(x_1, x_2, x_3, x_4) : x_1 = x_3 - x_2, \text{ and } x_4 = x_1 x_3\}$

4. (a) Let $A = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$. Show that $A^2 = 0$ and that $A + I_2$ is invertible.

   (b) Suppose that A is an $n \times n$ matrix with $A^2 = 0$. Must $A + I_n$ always be invertible? Either explain why or give a counter-example.

5. Find all the vectors in the kernel of each of the following linear transformations, and justify your answers.

   (a) The shear $T(\mathbf{x}) = \begin{bmatrix} 1 & -5 \\ 0 & 1 \end{bmatrix} \mathbf{x}$.

   (b) Reflection about a plane in $\mathbb{R}^3$.

   (c) $T : \mathbb{R}^2 \to \mathbb{R}^2$ given by projection onto the line $y = x$.

   (d) $T(\mathbf{x}) = A\mathbf{x}$, where A is an $n \times m$ matrix of rank m.

6. Given subspaces W_1 and W_2 of $\mathbb{R}^n$, set

$$W_1 + W_2 = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{x} = \mathbf{w_1} + \mathbf{w_2} \text{ for some } \mathbf{w_1} \in W_1 \text{ and } \mathbf{w_2} \in W_2\}.$$

Prove that $W_1 + W_2$ is a subspace of $\mathbb{R}^n$.

7. Let $V = \mathbb{R}^2$ and $S = \left\{ \begin{bmatrix} 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ -2 \end{bmatrix} \right\}$.

   (a) Show that $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is in Span(S). [Hint: this is the same as solving a certain system of equations]

   (b) Show that every vector $\begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$ in $\mathbb{R}^2$ is in Span(S).

8. Let S be a non-empty subset of $\mathbb{R}^n$. Assume that each vector in Span(S) can be written in one and only one way as a linear combination of vectors in S. Show that S is linearly independent.