Carleton College, winter 2014
Math 232, Solutions to practice midterm 2
Prof. Jones

1. Find a bases for the kernel and image of the following matrix:

$$A = \begin{bmatrix} 1 & 0 & 2 & 0 & 1 \\ -3 & 0 & 0 & 1 & 0 \\ -1 & 0 & 4 & 1 & 2 \end{bmatrix}.$$

To find a basis for the kernel, we need to row reduce the augmented matrix

$$\left[\begin{array}{ccccc|c} 1 & 0 & 2 & 0 & 1 & 0 \\ -3 & 0 & 0 & 1 & 0 & 0 \\ -1 & 0 & 4 & 1 & 2 & 0 \end{array}\right],$$

and we get the reduced row-echelon form

$$\left[\begin{array}{ccccc|c} 1 & 0 & 0 & -1/3 & 0 & 0 \\ 0 & 0 & 1 & 1/6 & 1/2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right],$$

Thus the free variables are x_2, x_4, and x_5, and we set $x_2 = t_1$, $x_4 = t_2$, and $x_5 = t_3$. The set of solutions is then

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} & \frac{1}{3}t_2 & \\ t_1 & & \\ & -\frac{1}{6}t_2 & -\frac{1}{2}t_3 \\ & t_2 & \\ & & t_3 \end{bmatrix}.$$

Separating the arbitrary constants t_1, t_2, t_3 into their own vectors then yields a basis of

$$\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1/3 \\ 0 \\ -1/6 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ -1/2 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

As for the image, the above reduced row-echelon form shows that the non-redundant columns are the first and the third, so a basis for the image is

$$\left\{ \begin{bmatrix} 1 \\ -3 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 4 \end{bmatrix} \right\}.$$

2. Find a basis for the following subspace of P_4.

$$W = \{p(x) \in P_4 \mid p(1) = p(-1) = 0\}.$$

What is the dimension of W?

Note that W is the same as

$$\{a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0 \in P_4 \mid a_4 + a_3 + a_2 + a_1 + a_0 = 0, a_4 - a_3 + a_2 - a_1 + a_0 = 0\}.$$

There are different ways to do this problem, but the most systematic is to find all solutions to the system of equations

$$\begin{array}{cccccc} a_0 & +a_1 & +a_2 & +a_3 & +a_4 & = 0 \\ a_0 & -a_1 & +a_2 & -a_3 & +a_4 & = 0 \end{array}.$$

So we put the augmented matrix

$$\left[\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & -1 & 1 & -1 & 1 & 0 \end{array}\right]$$

in reduced row-echelon form. We get

$$\left[\begin{array}{ccccc|c} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \end{array}\right]$$

Thus our free variables are a_2, a_3, and a_4. Set $a_2 = t_1$, $a_3 = t_2$, and $a_4 = t_3$. The set of solutions is then

$$\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} = \begin{bmatrix} -t_1 & & -t_3 \\ & -t_2 & \\ t_1 & & \\ & t_2 & \\ & & t_3 \end{bmatrix}.$$

Separating the arbitrary constants t_1, t_2, t_3 into their own vectors then yields a basis of

$$\left\{ \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

But we need to re-convert these to polynomials, since W is a subspace of P_4, and hence any basis for it should consist of elements of P_4. So a basis for W is

$$\{-1 + x^2, -x + x^3, -1 + x^4\},$$

and W has dimension 3.

3. Determine whether the following mappings are linear transformations. Either prove that a given map is linear or give a counterexample to show it's not linear.

(a) $T : \mathbb{R}^2 \to \mathbb{R}^3$ defined by $T((x_1, x_2)) = (2x_1, x_1 + 4, 5x_2)$

Not linear since $2T((0,0)) = 2(0,4,0) = (0,8,0)$ but $T(2(0,0)) = T((0,0)) = (0,4,0)$.

(b) $T : P_2 \to P_3$ defined by $T(a_2x^2 + a_1x + a_0) = a_0x^3 + (a_1 - a_0)x^2 + 3a_2 - (1/2)a_0$

Linear. Let $\vec{u} = u_2x^2 + u_1x + u_0$ and $\vec{v} = v_2x^2 + v_1x + v_0$. Then

$$\begin{aligned} T(\vec{u}+\vec{v}) &= (u_0+v_0)x^3 + (u_1+v_1-u_0-v_0)x^2 + 3(u_2+v_2) - (1/2)(u_0+v_0) \\ &= (u_0x^3 + (u_1-u_0)x^2 + 3u_2 - (1/2)u_0) + (v_0x^3 + (v_1-v_0)x^2 + 3v_2 - (1/2)v_0) \\ &= T(\vec{u}) + T(\vec{v}) \end{aligned}$$

A similar calculation shows that $T(c\vec{u}) = cT(\vec{u})$.

4. Let V be a subspace of $\mathbb{R}^n$ with $\dim(V) = n$. Explain why $V = \mathbb{R}^n$.

Since $\dim(V) = n$, V has a basis B containing n elements. Since B is a basis for V, B is linearly independent. However, n linearly independent vectors in the n-dimensional space $\mathbb{R}^n$ must form a basis for $\mathbb{R}^n$, by Theorem 3.3.4(c). Thus B is a basis for both V and $\mathbb{R}^n$, and so $\text{Span}(B) = V$ and $\text{Span}(B) = \mathbb{R}^n$. Hence $V = \mathbb{R}^n$.

5. (a) Consider the mapping $T : \mathbb{R}^{2\times 2} \to \mathbb{R}^{2\times 2}$ defined by

$$T\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a+b & c-b \\ b+2d-3c & d+4a \end{pmatrix}.$$

Prove that T is a linear transformation.

Let

$$\vec{u} = \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix} \qquad \vec{v} = \begin{pmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{pmatrix}$$

Then

$$\begin{aligned} T(\vec{u}+\vec{v}) &= \begin{pmatrix} u_{11}+v_{11}+u_{12}+v_{12} & u_{21}+v_{21}-(u_{12}+v_{12}) \\ u_{12}+v_{12}+2(u_{22}+v_{22})-3(u_{21}+v_{21}) & u_{22}+v_{22}+4(u_{11}+v_{11}) \end{pmatrix} \\ &= \begin{pmatrix} u_{11}+u_{12} & u_{21}-u_{12} \\ u_{12}+2u_{22}-3u_{21} & u_{22}+4u_{11} \end{pmatrix} + \begin{pmatrix} v_{11}+v_{12} & v_{21}-v_{12} \\ v_{12}+2v_{22}-3v_{21} & v_{22}+4v_{11} \end{pmatrix} \\ &= T(\vec{u}) + T(\vec{v}) \end{aligned}$$

A similar calculation shows that $T(c\vec{u}) = cT(\vec{u})$.

(b) Given the basis $\alpha = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$ of $\mathbb{R}^{2\times 2}$, give the matrix $[T]_\alpha$ of T with respect to the basis α.

$$T\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix} = 1\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + 0\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + 0\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + 4\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

Thus the first column of $[T]_\alpha$ is

$$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 4 \end{bmatrix}.$$

Proceeding similarly with the other columns, we get

$$[T]_\alpha = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & -3 & 2 \\ 4 & 0 & 0 & 1 \end{bmatrix}$$

(c) Show that T is an isomorphism, and use the determinant in your solution.

To show T is an isomorphism, one method is to show directly that T is invertible. We can accomplish this by showing that $[T]_\alpha$ from part (b) is an invertible matrix. This, in turn, we can do by finding the determinant of $[T]_\alpha$. Expand along the first column to get

$$\begin{aligned} \det[T]_\alpha &= \det\begin{bmatrix} -1 & 1 & 0 \\ 1 & -3 & 2 \\ 0 & 0 & 1 \end{bmatrix} - 4\det\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & -3 & 2 \end{bmatrix} \\ &= \det\begin{bmatrix} -1 & 1 \\ 1 & -3 \end{bmatrix} - 4(1\cdot 1\cdot 2) \\ &= (3-1) - 8 = -6 \end{aligned}$$

Since this determinant is non-zero, the matrix is invertible, and thus T is an isomorphism.

6. The mapping $T : \mathbb{R}^2 \to P_2$ given by $T\left(\begin{bmatrix} a_1 \\ a_2 \end{bmatrix}\right) = (a_1 + a_2)x^2 + a_2 x + a_1$ is a linear transformation.

(a) Prove that $\alpha = \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \end{bmatrix} \right\}$ is a basis for $\mathbb{R}^2$ and $\beta = \{x^2 + 2, x^2 + x, 1\}$ is a basis for P_2.

First note that α is a basis for $\mathbb{R}^2$ since α is linearly independent (it is a set of two vectors and neither is a multiple of the other) and the dimension of $\mathbb{R}^2$ is two. Similarly, one can check that β is linearly independent by assuming that $c_1(x^2+2)+c_2(x^2+x)+c_3(1)=\vec{0}$, setting up a system of equations involving c_1, c_2, and c_3, and showing that $c_1=c_2=c_3=0$. Since $P_2(\mathbb{R})$ has dimension 3 this shows that β is a basis, by Theorem 3.3.4(c).

(b) Find the matrix $[T]_\alpha^\beta$

To compute the first column of $[T]_\alpha^\beta$, we find $T((1,2))$ and write it in β-coordinates. We have

$$T((1,2)) = 3x^2+2x+1 = 1(x^2+2)+2(x^2+x)-1(1)$$

and thus the first column of $[T]_\alpha^\beta$ is

$$\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}.$$

Note that to find the β-coordinates of $3x^2+2x+1$, you can either eyeball a solution as follows: only x^2+x involves x, and so we must have $2(x^2+x)$, and then the only other term with x^2 is x^2+2, so we must add on $1(x^2+2)$ to get $3x^2$, then we have a constant term of 2, so we must subtract 1(1). Or you can take the more systematic approach of writing

$$3x^2+2x+1 = a_1(x^2+2)+a_2(x^2+x)+a_3(1),$$

which leads to the system of equations

$$\begin{array}{rrrl} & a_2 & +a_1 & =3 \\ & a_2 & & =2 \\ a_3 & & +2a_1 & =1 \end{array}$$

which you can solve.

In the same way we can find the second column of $[T]_\alpha^\beta$, which gives

$$[T]_\alpha^\beta = \begin{bmatrix} 1 & -1 \\ 2 & 0 \\ -1 & 1 \end{bmatrix}$$

(c) What is the dimension of Ker(T)? Find a basis for Ker(T).

We put the augmented matrix

$$\left[\begin{array}{rr|r} 1 & -1 & 0 \\ 2 & 0 & 0 \\ -1 & 1 & 0 \end{array}\right]$$

In reduced row-echelon form, which gives

$$\left[\begin{array}{rr|r} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

There are no free variables, and hence $\ker T = \{\vec{0}\}$. So $\ker T$ is zero-dimensional, and has basis $\{\vec{0}\}$.

(d) Since the dimension of $\ker T$ is zero, by the rank-nullity theorem we know that the dimension of the image of T must equal the dimension of $\mathbb{R}^2$, which is two. A basis of the image of the matrix $[T]_\alpha^\beta$ is

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

To turn this into a basis for the image of T, we need to re-convert these vectors (which are in β-coordinates) into elements of P_2. Doing so gives the two elements

$$1 \cdot (x^2 + 2) + 2 \cdot (x^2 + x) + (-1) \cdot 1 = 3x^2 + 2x + 1$$

$$-1 \cdot (x^2 + 2) + 0 \cdot (x^2 + x) + (1) \cdot 1 = -x^2 - 1$$

So $\{3x^2 + 2x + 1, -x^2 - 1\}$ is a basis for the image of T.