Carleton College, Winter 2014
Math 232, Practice Midterm 2
Prof. Jones

Note: the exam will have 3-4 questions like the ones below, as well as a section of short answer questions similar to the one on the first exam. To review for those, the best thing is to work the true/false review questions listed on the main course webpage.

1. Find a bases for the kernel and image of the following matrix:
$$A = \begin{bmatrix} 1 & 0 & 2 & 0 & 1 \\ -3 & 0 & 0 & 1 & 0 \\ -1 & 0 & 4 & 1 & 2 \end{bmatrix}.$$

2. Find a basis for the following subspace of P_4.
$$W = \{p(x) \in P_4 \mid p(1) = p(-1) = 0\}.$$
What is the dimension of W?

3. Determine whether the following mappings are linear transformations. Either prove that a given map is linear or give a counterexample to show it's not linear.
(a) $T : \mathbb{R}^2 \to \mathbb{R}^3$ defined by $T((x_1, x_2)) = (2x_1, x_1 + 4, 5x_2)$
(b) $T : P_2 \to P_3$ defined by $T(a_2x^2 + a_1x + a_0) = a_0x^3 + (a_1 - a_0)x^2 + 3a_2 - (1/2)a_0$

4. Let V be a subspace of $\mathbb{R}^n$ with $\dim(V) = n$. Explain why $V = \mathbb{R}^n$.

5. (a) Consider the mapping $T : \mathbb{R}^{2\times 2} \to \mathbb{R}^{2\times 2}$ defined by
$$T\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a+b & c-b \\ b+2d-3c & d+4a \end{pmatrix}.$$
Prove that T is a linear transformation.
(b) Given the basis $\alpha = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$ of $\mathbb{R}^{2\times 2}$,
give the matrix $[T]_\alpha$ of T with respect to the basis α.
(c) Show that T is an isomorphism, and use the determinant in your solution.

6. The mapping $T : \mathbb{R}^2 \to P_2$ given by $T\left(\begin{bmatrix} a_1 \\ a_2 \end{bmatrix}\right) = (a_1 + a_2)x^2 + a_2x + a_1$ is a linear transformation.
(a) Prove that $\alpha = \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \end{bmatrix} \right\}$ is a basis for $\mathbb{R}^2$ and $\beta = \{x^2 + 2, x^2 + x, 1\}$ is a basis for P_2.
(b) Find the matrix $[T]_\alpha^\beta$
(c) What is the dimension of $\mathrm{Ker}(T)$? Find a basis for $\mathrm{Ker}(T)$.
(d) What is the dimension of $\mathrm{Im}(T)$? Find a basis for $\mathrm{Im}(T)$