

**Carleton College, winter 2014**  
**Math 232, Solutions to practice midterm 2**  
**Prof. Jones**

1. Find a bases for the kernel and image of the following matrix:

$$A = \begin{bmatrix} 1 & 0 & 2 & 0 & 1 \\ -3 & 0 & 0 & 1 & 0 \\ -1 & 0 & 4 & 1 & 2 \end{bmatrix}.$$

To find a basis for the kernel, we need to row reduce the augmented matrix

$$\left[ \begin{array}{ccccc|c} 1 & 0 & 2 & 0 & 1 & 0 \\ -3 & 0 & 0 & 1 & 0 & 0 \\ -1 & 0 & 4 & 1 & 2 & 0 \end{array} \right],$$

and we get the reduced row-echelon form

$$\left[ \begin{array}{ccccc|c} 1 & 0 & 0 & -1/3 & 0 & 0 \\ 0 & 0 & 1 & 1/6 & 1/2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right],$$

Thus the free variables are  $x_2, x_4$ , and  $x_5$ , and we set  $x_2 = t_1$ ,  $x_4 = t_2$ , and  $x_5 = t_3$ . The set of solutions is then

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} \frac{1}{3}t_2 \\ t_1 \\ -\frac{1}{6}t_2 - \frac{1}{2}t_3 \\ t_2 \\ t_3 \end{bmatrix}.$$

Separating the arbitrary constants  $t_1, t_2, t_3$  into their own vectors then yields a basis of

$$\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1/3 \\ 0 \\ -1/6 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ -1/2 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

As for the image, the above reduced row-echelon form shows that the non-redundant columns are the first and the third, so a basis for the image is

$$\left\{ \begin{bmatrix} 1 \\ -3 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 4 \end{bmatrix} \right\}.$$

2. Find a basis for the following subspace of  $P_4$ .

$$W = \{p(x) \in P_4 \mid p(1) = p(-1) = 0\}.$$

What is the dimension of  $W$ ?

Note that  $W$  is the same as

$$\{a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0 \in P_4 \mid a_4 + a_3 + a_2 + a_1 + a_0 = 0, a_4 - a_3 + a_2 - a_1 + a_0 = 0\}.$$

There are different ways to do this problem, but the most systematic is to find all solutions to the system of equations

$$\begin{array}{cccccc} a_0 & +a_1 & +a_2 & +a_3 & +a_4 & = 0 \\ a_0 & -a_1 & +a_2 & -a_3 & +a_4 & = 0 \end{array} .$$

So we put the augmented matrix

$$\left[ \begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & -1 & 1 & -1 & 1 & 0 \end{array} \right]$$

in reduced row-echelon form. We get

$$\left[ \begin{array}{ccccc|c} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \end{array} \right]$$

Thus our free variables are  $a_2$ ,  $a_3$ , and  $a_4$ . Set  $a_2 = t_1$ ,  $a_3 = t_2$ , and  $a_4 = t_3$ . The set of solutions is then

$$\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} = \begin{bmatrix} -t_1 & & -t_3 \\ & -t_2 & \\ t_1 & & \\ & t_2 & \\ & & t_3 \end{bmatrix} .$$

Separating the arbitrary constants  $t_1, t_2, t_3$  into their own vectors then yields a basis of

$$\left\{ \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\} .$$

But we need to re-convert these to polynomials, since  $W$  is a subspace of  $P_4$ , and hence any basis for it should consist of elements of  $P_4$ . So a basis for  $W$  is

$$\{-1 + x^2, -x + x^3, -1 + x^4\},$$

and  $W$  has dimension 3.

3. Determine whether the following mappings are linear transformations. Either prove that a given map is linear or give a counterexample to show it's not linear.

(a)  $T : \mathbb{R}^2 \rightarrow \mathbb{R}^3$  defined by  $T((x_1, x_2)) = (2x_1, x_1 + 4, 5x_2)$

Not linear since  $2T((0, 0)) = 2(0, 4, 0) = (0, 8, 0)$  but  $T(2(0, 0)) = T((0, 0)) = (0, 4, 0)$ .

(b)  $T : P_2 \rightarrow P_3$  defined by  $T(a_2x^2 + a_1x + a_0) = a_0x^3 + (a_1 - a_0)x^2 + 3a_2 - (1/2)a_0$

Linear. Let  $\vec{u} = u_2x^2 + u_1x + u_0$  and  $\vec{v} = v_2x^2 + v_1x + v_0$ . Then

$$\begin{aligned} T(\vec{u} + \vec{v}) &= (u_0 + v_0)x^3 + (u_1 + v_1 - u_0 - v_0)x^2 + 3(u_2 + v_2) - (1/2)(u_0 + v_0) \\ &= (u_0x^3 + (u_1 - u_0)x^2 + 3u_2 - (1/2)u_0) + (v_0x^3 + (v_1 - v_0)x^2 + 3v_2 - (1/2)v_0) \\ &= T(\vec{u}) + T(\vec{v}) \end{aligned}$$

A similar calculation shows that  $T(c\vec{u}) = cT(\vec{u})$ .

4. Let  $V$  be a linear space of dimension  $n$ , and let  $S$  be a linearly independent subset of  $V$ . Suppose that  $S'$  is a proper subset of  $S$  (this means that  $S'$  is contained in  $S$  and  $S' \neq S$ ). Prove that  $S'$  cannot be a basis for  $V$ . [Hint: use Theorem 3.3.4.]

Since  $S$  is a linearly independent subset of  $V$ , by Theorem 3.3.4(a),  $S$  can contain at most  $n$  vectors. Since  $S'$  is a proper subset of  $S$ , it must contain strictly fewer than  $n$  vectors. But by Theorem 3.3.4(b), at least  $n$  vectors are required to span  $V$ , and hence  $S'$  cannot span  $V$ . Therefore  $S'$  cannot be a basis for  $V$ .

5. (a) Consider the mapping  $T : \mathbb{R}^{2 \times 2} \rightarrow \mathbb{R}^{2 \times 2}$  defined by

$$T \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a + b & c - b \\ b + 2d - 3c & d + 4a \end{pmatrix}.$$

Prove that  $T$  is a linear transformation.

Let

$$\vec{u} = \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix} \quad \vec{v} = \begin{pmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{pmatrix}$$

Then

$$\begin{aligned} T(\vec{u} + \vec{v}) &= \begin{pmatrix} u_{11} + v_{11} + u_{12} + v_{12} & u_{21} + v_{21} - (u_{12} + v_{12}) \\ u_{12} + v_{12} + 2(u_{22} + v_{22}) - 3(u_{21} + v_{21}) & u_{22} + v_{22} + 4(u_{11} + v_{11}) \end{pmatrix} \\ &= \begin{pmatrix} u_{11} + u_{12} & u_{21} - u_{12} \\ u_{12} + 2u_{22} - 3u_{21} & u_{22} + 4u_{11} \end{pmatrix} + \begin{pmatrix} v_{11} + v_{12} & v_{21} - v_{12} \\ v_{12} + 2v_{22} - 3v_{21} & v_{22} + 4v_{11} \end{pmatrix} \\ &= T(\vec{u}) + T(\vec{v}) \end{aligned}$$

A similar calculation shows that  $T(c\vec{u}) = cT(\vec{u})$ .

- (b) Given the basis  $\alpha = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$  of  $\mathbb{R}^{2 \times 2}$ ,  
 give the matrix  $[T]_\alpha$  of  $T$  with respect to the basis  $\alpha$ .

$$T \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix} = 1 \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + 0 \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + 0 \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + 4 \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

Thus the first column of  $[T]_\alpha$  is

$$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 4 \end{bmatrix}.$$

Proceeding similarly with the other columns, we get

$$[T]_\alpha = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & -3 & 2 \\ 4 & 0 & 0 & 1 \end{pmatrix}$$

6. The mapping  $T : \mathbb{R}^2 \rightarrow P_2$  given by  $T \left( \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \right) = (a_1 + a_2)x^2 + a_2x + a_1$  is a linear transformation.

- (a) Prove that  $\alpha = \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \end{bmatrix} \right\}$  is a basis for  $\mathbb{R}^2$  and  $\beta = \{x^2 + 2, x^2 + x, 1\}$  is a basis for  $P_2$ .

First note that  $\alpha$  is a basis for  $\mathbb{R}^2$  since  $\alpha$  is linearly independent (it is a set of two vectors and neither is a multiple of the other) and the dimension of  $\mathbb{R}^2$  is two. Similarly, one can check that  $\beta$  is linearly independent by assuming that  $c_1(x^2 + 2) + c_2(x^2 + x) + c_3(1) = \vec{0}$ , setting up a system of equations involving  $c_1, c_2$ , and  $c_3$ , and showing that  $c_1 = c_2 = c_3 = 0$ . Since  $P_2(\mathbb{R})$  has dimension 3 this shows that  $\beta$  is a basis, by Theorem 3.3.4(c).

- (b) Find the matrix  $[T]_\alpha^\beta$

To compute the first column of  $[T]_\alpha^\beta$ , we find  $T((1, 2))$  and write it in  $\beta$ -coordinates. We have

$$T((1, 2)) = 3x^2 + 2x + 1 = 1(x^2 + 2) + 2(x^2 + x) - 1(1)$$

and thus the first column of  $[T]_\alpha^\beta$  is

$$\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}.$$

Note that to find the  $\beta$ -coordinates of  $3x^2 + 2x + 1$ , you can either eyeball a solution as follows: only  $x^2 + x$  involves  $x$ , and so we must have  $2(x^2 + x)$ , and then the only other term with  $x^2$  is  $x^2 + 2$ , so we must add on  $1(x^2 + 2)$  to get  $3x^2$ , then we have a constant term of 2, so we must subtract  $1(1)$ . Or you can take the more systematic approach of writing

$$3x^2 + 2x + 1 = a_1(x^2 + 2) + a_2(x^2 + x) + a_3(1),$$

which leads to the system of equations

$$\begin{array}{rcl} a_2 & +a_1 & = 3 \\ a_2 & & = 2 \\ a_3 & +2a_1 & = 1 \end{array}$$

which you can solve.

In the same way we can find the second column of  $[T]_{\alpha}^{\beta}$ , which gives

$$[T]_{\alpha}^{\beta} = \begin{bmatrix} 1 & -1 \\ 2 & 0 \\ -1 & 1 \end{bmatrix}$$

(c) What is the dimension of  $\text{Ker}(T)$ ? Find a basis for  $\text{Ker}(T)$ .

We put the augmented matrix

$$\left[ \begin{array}{cc|c} 1 & -1 & 0 \\ 2 & 0 & 0 \\ -1 & 1 & 0 \end{array} \right]$$

In reduced row-echelon form, which gives

$$\left[ \begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{array} \right]$$

There are no free variables, and hence  $\ker T = \{\vec{0}\}$ . So  $\ker T$  is zero-dimensional, and has basis  $\{\vec{0}\}$ .

(d) Since the dimension of  $\ker T$  is zero, by the rank-nullity theorem we know that the dimension of the image of  $T$  must equal the dimension of  $\mathbb{R}^2$ , which is two. A basis of the image of the matrix  $[T]_{\alpha}^{\beta}$  is

$$\left\{ \left[ \begin{array}{c} 1 \\ 2 \\ -1 \end{array} \right], \left[ \begin{array}{c} -1 \\ 0 \\ 1 \end{array} \right] \right\}.$$

To turn this into a basis for the image of  $T$ , we need to re-convert these vectors (which are in  $\beta$ -coordinates) into elements of  $P_2$ . Doing so gives the two elements

$$\begin{aligned} 1 \cdot (x^2 + 2) + 2 \cdot (x^2 + x) + (-1) \cdot 1 &= 3x^2 + 2x + 1 \\ -1 \cdot (x^2 + 2) + 0 \cdot (x^2 + x) + (1) \cdot 1 &= -x^2 - 1 \end{aligned}$$

So  $\{3x^2 + 2x + 1, -x^2 - 1\}$  is a basis for the image of  $T$ .