Math 232 – comments/answers for some final exam review problems

Section 1.1, #25. The system has solutions if $k = 7$. When $k = 7$, the the system has infinitely many solutions, which are given by

$$\vec{x} = \begin{bmatrix} 1-t \\ -3+2t \\ t \end{bmatrix}.$$

Section 2.2, #25. The matrix A represents a horizontal shear (shearing from left to right), and its inverse $A^{-1} = \begin{bmatrix} 1 & -k \\ 0 & 1 \end{bmatrix}$ represents another horizontal shear (shearing from right to left).

Section 3.2, #3. You can directly verify this by writing down two typical elements of W and using the definition of subspace. A quicker method is to notice that W is the image of the matrix $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$. The image of any matrix is a subspace, by Theorem 3.2.2.

Section 4.1, #5. A typical element of this subset is of the form $p(t) = bt$, and thus this is a subspace, with basis $\{t\}$.

Section 4.3, #13. A basis of the kernel of T is

$$\left\{ \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} \right\},$$

while a basis for the image of T is

$$\left\{ \begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix} \right\}.$$

Since the kernel is non-trivial, T is not an isomorphism.

Section 6.1, #17. The determinant of this matrix works out to $2(k-1)(k+1)$, so k cannot be 1 or -1.

Section 7.3, #13. The eigenvalues are $0, 1, -1$, and an eigenbasis is

$$\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -3 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} \right\}.$$

Section 7.4, #15. Diagonalizable. The eigenvalues are $-1, 1, 1$, and an eigenbasis is

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

From these you can find S and D.

Section 7.4, #47. The non-zero even functions, of the form $f(x) = a + cx^2$, are eigenfunctions with eigenvalue 1, and the non-zero odd functions, of the form $f(x) = bx$, are eigenfunctions with eigenvalue -1. Yes, T is diagonalizable, and the set $\{1, x, x^2\}$ is an eigenbasis, with corresponding eigenvalues $1, -1, 1$.

Ch. 1 T/F, #26. True. The equation $A\vec{x} = \vec{b}$ corresponds to a system of 4 equations in 3 unknowns. There will be some choice of $\vec{b}$ such that the resulting augmented matrix has a row that is all zeroes followed by a non-zero number, when put in reduced row-echelon form. Another way of thinking about this problem is that A corresponds to a linear transformation from $\mathbb{R}^3$ to $\mathbb{R}^4$, and the image of such a transformation (by the rank-nullity theorem) must have dimension at least 3. So there must be a vector $\vec{b}$ in $\mathbb{R}^4$ that is not in the image of A, that is, $A\vec{x} = \vec{b}$ has no solution.

Ch. 3 T/F, #37. False. A counterexample is to let V be the x-axis in $\mathbb{R}^2$ (which is a subspace since it a line through the origin), and W be the y-axis. Then $V \cup W$ is not closed under vector addition, and so is not a subspace.

Ch. 6 T/F, #37. True. If all the $(n-1) \times (n-1)$ sub-matrices failed to be invertible, then they all would have determinant zero, which means expanding along any row or column of A would force the determinant of A to be zero. But this is impossible since A is invertible.

Ch. 7 T/F, #25. False. A matrix can be diagonalizable while having an eigenvalue of zero, which forces it to be non-invertible. For instance, take any 2×2 matrix with eigenvalues 0 and 1. Such a matrix must be diagonalizable since it has two distinct eigenvalues, but it is not invertible. As another example, take the zero matrix. It's already diagonal, and so is trivially diagonalizable. But it's obviously not invertible.

Ch. 7 T/F, #39. True. $\det A$ is the constant term of the characteristic polynomial of A, and the trace of A is the negative of the linear term. Thus the characteristic polynomial of A is $\lambda^2 - \lambda - 6 = (\lambda - 3)(\lambda + 2)$. So A has two distinct eigenvalues, and hence is diagonalizable.