

**Carleton College, Spring 2018**  
**Math 232, Practice Midterm 1**  
**Prof. Jones**

1. True or False (short answer). Write a few sentences justifying your answer.
  - (a) Let  $A$  be an  $n \times n$  matrix of rank  $n$ , and  $\mathbf{b}$  is a vector in  $\mathbb{R}^n$ . Then there must be  $\mathbf{v} \in \mathbb{R}^n$  with  $A\mathbf{v} = \mathbf{b}$ .
  - (b) Suppose that  $A$  is an  $n \times n$  matrix with  $A^2 = 0$ . Then  $A - I_n$  must be invertible.
  - (c) If  $A$  is a  $2 \times 2$  matrix with  $A^2 = I_2$ , then  $A = I_2$  or  $A = -I_2$ .
  - (d) If  $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$  is a set of linearly independent vectors in  $\mathbb{R}^3$  and  $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$  is a linear transformation, then  $\{T(\vec{v}_1), T(\vec{v}_2), T(\vec{v}_3)\}$  is linearly independent.
2. Harry, who likes puzzles, tells you he is thinking of four numbers. Summing the first and fourth and twice the second yields zero. The third minus three times the first gives 1. Adding the third to twice the fourth and then adding four times the second, and then subtracting the first gives 1. You have a feeling that Harry has not given you enough information to nail down all four numbers, and to illustrate that you plan to present him with all the possibilities for the four numbers in question. What are these possibilities?
3. For each of the following subsets  $W$  of a vector space  $V$ , determine if  $W$  is a subspace of  $V$ . In each case either prove that  $W$  is a subspace or give a concrete reason why it is not a subspace.
  - (a)  $V = \mathbb{R}^4$ , and  $W = \{(x_1, x_2, x_3, x_4) \mid x_1 = x_3 - x_2, \text{ and } x_4 = 0\}$
  - (b)  $V = \mathbb{R}^4$ , and  $W = \{(x_1, x_2, x_3, x_4) \mid x_1 = x_3 - x_2, \text{ and } x_4 = x_1x_3\}$
4. Find all the vectors in the kernel of each of the following linear transformations, and justify your answers.
  - (a) The shear  $T(\mathbf{x}) = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix} \mathbf{x}$ , where  $k$  is a real number.
  - (b) Reflection about a plane in  $\mathbb{R}^3$ .
  - (c)  $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  given by projection onto the line  $y = x$ .
  - (d)  $T(\mathbf{x}) = A\mathbf{x}$ , where  $A$  is an  $n \times m$  matrix of rank  $m$ .
5. Given subspaces  $W_1$  and  $W_2$  of  $\mathbb{R}^n$ , show that the intersection  $W_1 \cap W_2$  is a subspace of  $\mathbb{R}^n$ . (The intersection is the set of vectors that belong to both subspaces.)

6. Let  $S$  be a non-empty subset of  $\mathbb{R}^n$ . Assume that each vector in  $\text{Span}(S)$  can be written in one and only one way as a linear combination of vectors in  $S$ . Show that  $S$  is linearly independent.