Carleton College, Winter 2014
Math 121, Practice Midterm 2 solutions
Prof. Jones

Note: the exam will have a section of true-false questions, like the one below.

1. True or False. Briefly explain your answer. An incorrectly justified answer may not receive full (or any) credit.

   (a) Suppose that as a car brakes, its deceleration is proportional to the square of its velocity. Then its motion is described by the differential equation

   $$\frac{dv}{dt} = \frac{k}{v^2},$$

   where $v(t)$ is the car's velocity at time t and k is a constant.

   False. The correct DE would be $\frac{dv}{dt} = kv^2$.

   (b) In part (a), the constant k is positive.

   False. Since the car is decelerating, its velocity is decreasing, and so dv/dt is negative. However, since it is still moving forwards, $v(t)$ is positive. This forces k to be negative.

   (c) Suppose that for the series $\sum_{n=1}^{\infty} a_n$, the sequence of terms a_n satisfies $\lim_{n\to\infty} a_n = 0$. Then $\sum_{n=1}^{\infty} a_n$ might converge, or might diverge; there is not enough information to tell.

   True. There are series with $\lim_{n\to\infty} a_n = 0$ that converge (for instance, $a_n = (1/2)^n$ gives a convergent geometric series) and also ones that diverge (for instance, $a_n = 1/n$ gives the harmonic series, which diverges). So there is not enough information to tell.

   (d) If $f(x)$ is a function with $f(0) = 0$, $f'(0) = 1$ and $f''(0) = -2$, then its second Taylor polynomial at $a = 0$ is $T_2(x) = x - 2x^2$.

   False. The correct Taylor polynomial is $T_2(x) = x - x^2$.

   (e) Let $T_2(x)$ and $T_4(x)$ be the degree 2 and 4 Taylor polynomials centered at $a = 0$ for a function $f(x)$. Then $T_4(2)$ is always a better approximation to $f(2)$ than $T_2(2)$.

   False. In class we saw that for the function $f(x) = \ln(x+1)$, we had $T_2(2) = 0$ and $T_4(2) = -4/3$. But the real value of $f(2)$ is $\ln(3)$, which is a little more than 1.

2. For $f(x) = xe^x$, find the fourth Taylor polynomial $T_4(x)$ at $a = 0$. Then use it to approximate $-\frac{1}{e}$.

   Solution: We have

   $$\begin{aligned} f'(x) &= xe^x + e^x = (x+1)e^x \\ f''(x) &= (x+1)e^x + e^x = (x+2)e^x \\ f^{(3)}(x) &= (x+2)e^x + e^x = (x+3)e^x \\ f^{(4)}(x) &= (x+3)e^x + e^x = (x+4)e^x, \end{aligned}$$

and so

$$f(0) = 0, \quad f'(0) = 1, \quad f''(0) = 2, \quad f^{(3)}(0) = 3, \quad f^{(4)}(0) = 4.$$

Thus

$$T_4(x) = 0 + 1 \cdot x + \frac{2}{2!}x^2 + \frac{3}{3!}x^3 + \frac{4}{4!}x^4 = x + x^2 + \frac{1}{2}x^3 + \frac{1}{6}x^4.$$

Now we know that $T_4(x)$ gives an approximation to $f(x)$, and here $f(-1) = (-1)e^{-1} = -1/e$. So

$$-\frac{1}{e} \approx T_4(-1) = -1 + (-1)^2 + \frac{1}{2}(-1)^3 + \frac{1}{6}(-1)^4 = -1 + 1 - \frac{1}{2} + \frac{1}{6} = -\frac{1}{3}.$$

3. Solve the initial value problem

$$\frac{dy}{dx} = \frac{y}{x}, \qquad y(1) = 3$$

Solution: Separating variables gives $\frac{1}{y}\,dy = \frac{1}{x}\,dx$, and then integrating gives

$$\ln y = \ln x + C.$$

Exponentiating both sides now yields $y = e^{\ln x + C}$, which is the same as $y = e^{\ln x}e^C$, or $y = e^C x$. Replacing e^C by C then gives $y = Cx$. Since $y(1) = 3$, we have $3 = C$, and so $y = 3x$ is the final answer.

4. Solve the initial value problem

$$\frac{dy}{dx} = \frac{y^2}{x^2 + 1}, \qquad y(0) = 1$$

Solution: Separating variables gives $y^{-2}\,dy = \frac{1}{x^2+1}\,dx$. Integrating then gives $-\frac{1}{y} = \tan^{-1}(x) + C$. Thus we get

$$y = -\frac{1}{\tan^{-1}(x) + C}.$$

Since $y(0) = 1$, we get $1 = -1/C$, or $C = -1$. So $y = -1/(\tan^{-1}(x) - 1)$ is the answer.

5. Determine whether these series converge. If a series converges and is geometric, find its sum.

a) $\displaystyle\sum_{n=1}^{\infty} (2 + (-1)^n)$

Solution: This series is not geometric. Its terms are given by the sequence $a_n = 2 + (-1)^n$, and so a_n oscillates between 1 and 3. Hence the sequence of terms does not converge, and so certainly does not converge to zero. Therefore by the nth term test for divergence, this series must diverge.

b) $\displaystyle\sum_{n=1}^{\infty} \left(\frac{\pi}{e^n}\right)$

Solution: This series *is* geometric, as you can see by noting that the sequence of terms is $\pi/e, \pi/e^2, \pi/e^3, \pi/e^4, \ldots$, and so there is a common ratio between each term and the previous one: to get each term from the previous one, you multiply by $1/e$. So $r = 1/e$. Since $|r| < 1$, the series converges, and since the first term is π/e, the sum of the series is

$$\frac{\pi/e}{1 - \frac{1}{e}} = \frac{\pi}{e - 1}.$$

c) $\frac{8}{9} - \frac{16}{27} + \frac{32}{81} - \frac{64}{243} + \cdots$

Solution: This series is also geometric, since the ratio of each term to the previous term is $-2/3$. Since $r = -2/3$, we have $|r| < 1$, so the series converges. The sum is

$$\frac{8/9}{1 + \frac{2}{3}} = \frac{8}{15}.$$