Carleton College, Winter 2014
Math 121, Practice Final
Prof. Jones

Note: the exam will have a section of true-false questions, like the one below.

1. True or False. Briefly explain your answer. An incorrectly justified answer may not receive full (or any) credit.

   (a) It's possible for a power series to converge for all x in $(1, 2]$ and for $x = 0$ but not for any other value of x.

   This is false. Every power series has a radius of convergence, and so the set of x-values where the series converges will be an interval (though one that may or may not include its endpoints). The set given in this problem is not an interval.

   (b) Denote by $g(x)$ the twentieth derivative of $f(x) = xe^{-x^4}$. Then $g(0) = 0$.

   A power series for $f(x)$ can be found by substitution and multiplication by x. We have that

$$e^{-x^4} = 1+(-x^4)+\frac{(-x^4)^2}{2!}+\frac{(-x^4)^3}{3!}+\frac{(-x^4)^4}{4!}+\frac{(-x^4)^5}{5!}+\cdots = 1-x^4+\frac{x^8}{2}-\frac{x^{12}}{6}+\frac{x^{16}}{24}-\frac{x^{20}}{120}+\cdots$$

   So then

$$xe^{-x^4} = x - x^5 + \frac{x^9}{2} - \frac{x^{13}}{6} + \frac{x^{17}}{24} - \frac{x^{21}}{120} + \cdots$$

   Since this is a power series representation for $f(x)$, it is also the Taylor series for xe^{-x^4}. This means that the coefficient of the x^{20} term is the twentieth derivative of $f(x)$ evaluated at $x = 0$, divided by 20!. This is the same as $g(0)/20!$. But the x^{20} term is zero in our power series, so we conclude that $g(0) = 0$.

   (c) Suppose that for the series $\sum_{n=1}^{\infty} a_n$, the sequence of terms a_n satisfies $\lim_{n\to\infty} a_n = 0$. Then $\sum_{n=1}^{\infty} a_n$ might converge, or might diverge; there is not enough information to tell.

   This is true. The nth term test for divergence only tells us that if $\lim_{n\to\infty} a_n$ is *not* 0, then $\sum_{n=1}^{\infty} a_n$ diverges. When $\lim_{n\to\infty} a_n = 0$, we can't conclude anything. For instance, the series $\sum_{n=1}^{\infty} \frac{1}{n}$ and $\sum_{n=1}^{\infty} \frac{1}{n^2}$ both satisfy $\lim_{n\to\infty} a_n = 0$, yet the first diverges and the second converges.

   (d) To evaluate $\int \frac{2x^2-2}{x^3-3x}\, dx$, you must use partial fractions.

   False. You can use the substitution $u = x^3 - 3x$. Then $du = 3x^2 - 3\, dx = (2/3)(2x^2 - 2)\, dx$.

   (e) The Taylor series of any function $f(x)$ at $x = 0$ converges for all values of x.

   False. For instance, the Taylor series for $\ln(1 + x)$ at $x = 0$ converges only for x in the interval $(-1, 1]$.

2. Find the following integrals:

   (a) $\int \ln x\, dx$

Use Integration by parts: set $u = \ln x$ and $dv = dx$. Thus $du = 1/x\ dx$ and $v = x$. The integral then becomes $x \ln x - \int dx$, which is $x \ln x - x + C$.

(b) $\int_1^\infty \frac{\ln x}{x}\ dx$

This integral diverges. You can show this by comparison to $\int_1^\infty \frac{1}{x}\ dx$, or you can calculate it directly. To calculate directly, first let's find $\int \frac{\ln x}{x}\ dx$, which we can do using the substitution $u = \ln x$. Then $du = (1/x)\ dx$, so the integral becomes $\int u\ du$. This gives $u^2/2 + C$, or $(\ln x)^2/2 + C$. So now we have

$$\int_1^\infty \frac{\ln x}{x}\ dx = \lim_{t\to\infty} \left[(\ln x)^2/2\right]_1^t = \lim_{t\to\infty} (\ln t)^2/2 - 0,$$

and this last limit is infinity, meaning that the integral diverges.

(c) $\int \frac{x^3}{\sqrt{1-x^2}}\ dx$

This one is readymade for a trig substitution. Use $x = \sin\theta$. Then $dx = \cos\theta\ d\theta$, and we get

$$\int \frac{\sin^3\theta}{\sqrt{1-\sin^2\theta}} \cos\theta\ d\theta = \int \frac{\sin^3\theta}{\cos\theta}\cos\theta\ d\theta = \int \sin^3\theta\ d\theta.$$

Since the integral is of an odd power of sine, we convert all but one power to cosines, and then substitute $u = \cos\theta$:

$$\int \sin^3\theta\ d\theta = \int (1-\cos^2\theta)\sin\theta\ d\theta = -\int (1-u^2)\ du.$$

So we now get

$$\frac{1}{3}u^3 - u + C,$$

or

$$\frac{1}{3}\cos^3\theta - \cos\theta + C.$$

Since $x = \sin\theta$, by drawing our triangle (or using $\sin^2\theta + \cos^2\theta = 1$), we get $\cos\theta = \sqrt{1-x^2}$, and so our final answer is

$$\frac{1}{3}(1-x^2)^{3/2} - (1-x^2)^{1/2} + C.$$

3. Solve the initial value problem

$$\frac{dy}{dx} = \frac{y^2}{x^2+1}, \qquad y(0) = 1$$

Separate to get $y^{-2}dy = (x^2+1)\ dx$. Then integrate to get $-1/y = x^3/3 + x + C$. Solve for y to get

$$y = -\frac{1}{x^3/3 + x + C}.$$

Since $y(0) = 1$, we have $1 = -1/C$, so $C = -1$, and the final answer is $y = -\frac{1}{x^3/3+x-1}$.

4. The integral $\int_0^1 e^{-x^3}\,dx$ cannot be evaluated exactly. Use a method from the course this term to approximate this integral *to within an error of at most 1/60*. Leave your answer as a fraction. (So select a method where finding the error bound won't be to difficult)

   The most convenient error estimate we saw this term is the one for alternating series. So wouldn't it be great if we could express this integral with an alternating series. Well we can! Substitute $-x^3$ into the Taylor series for e^x at $x = 0$, and we get

$$e^{-x^3} = 1 - x^3 + \frac{x^6}{2!} - \frac{x^9}{3!} + \frac{x^{12}}{4!} - \cdots,$$

   and this equality holds for all x. Therefore

$$\int e^{-x^3}\,dx = x - \frac{x^4}{4} + \frac{x^7}{7\cdot 2!} - \frac{x^10}{10\cdot 3!} + \frac{x^{13}}{13\cdot 4!} - \cdots,$$

   And so

$$\begin{aligned}\int_0^1 e^{-x^3}\,dx &= \left(1 - \frac{1}{4} + \frac{1}{7\cdot 2!} - \frac{1}{10\cdot 3!} + \frac{1}{13\cdot 4!} - \cdots\right) - 0\\ &= 1 - \frac{1}{4} + \frac{1}{14} - \frac{1}{60} + \frac{1}{312} - \cdots\end{aligned}$$

   So we've expressed the definite integral we want as an alternating series. The alternating series error estimate then tells us that

$$\int_0^1 e^{-x^3}\,dx = 1 - \frac{1}{4} + \frac{1}{14},$$

   with an error of at most $1/60$. We we express $1 - 1/4 + 1/14$ as a fraction, we get $23/28$.

5. Describe another method for approximating the integral from the previous problem, and write down (but do not evaluate) a sum of five numbers that is an approximation of this integral.

   Another way to approximate the integral is to use Simpson's rule. When we use it with four sub-intervals (or in other words $n = 4$), we get an approximation of

$$\frac{1}{12}\left(e^0 + 4e^{(-1/4)^3} + 2e^{(-1/2)^3} + 4e^{(-3/4)^3} + e^{-1}\right).$$

   The error estimate here is much harder to find than in the previous problem.

6. Use ideas from the course to approximate e^2 to within 0.1. Leave you answer as a fraction, but explain why your approximations are correct to within 0.1.

   One way we can approximate e is to recognize that it's the value of the function $f(x) = e^x$ when $x = 1$. So we approximate the function e^x using its Taylor series at $x = 0$:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots,$$

and so if we truncate at $n = 4$, we get that the degree-4 Taylor polynomial for $f(x) = e^x$ at $x = 0$ is

$$1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24}.$$

So this tells us that

$$e = e^1 \approx 1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} = 65/24,$$

which is approximately 2.70833, though you don't need that for this problem). How good is this approximation? Well, we can use the Taylor series error estimate. Since $n = 4$ here, we need to take K to be a number such that $|f^{(5)}(x)| < K$ for x in $[0, 1]$. But $f^{(5)}(x) = e^x$, and $|e^x| \leq e$ for x in $[0, 1]$, and $e < 3$. So we take $K = 3$. The error estimate is

$$\frac{K|b-a|^{n+1}}{(n+1)!} = \frac{3 \cdot 1}{5!} = \frac{3}{120} = 0.025.$$

So we've found e to within the desired degree of accuracy. If you want to play more with this, find the error in the approximation to e that you get by plugging 1 into the degree-10 Taylor polynomial for e^x. (Note that you don't need to find the actual approximation in order to find the error, though of course you're welcome to find the approximation). Then do the same for the degree-100 Taylor polynomial.

7. Determine whether these series converge. If a series converges and is geometric, find its sum.

a) $\displaystyle\sum_{n=2}^{\infty} \frac{2^n}{n^2 - 1}$

Use the ratio test. The limit you get works out to 1/2, and so the series converges.

b) $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n^2 + 1}$

The expression for the nth term of this series is very close to $\sqrt{n}/n^2$ when n is large, and $\sqrt{n}/n^2 = n^{1/2}/n^2 = 1/n^{3/2}$. So use either direct (easier) or limit comparison to the convergent p-series $\sum_{n=1}^{\infty} 1/n^{3/2}$ to show that this series converges.

c) $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{n^n}$ [Hint: it's helpful to write $(n+1)^{n+1}$ as $(n+1)^n(n+1)$.]

This one is a good candidate for the ratio test, since it involves an $n!$. When you apply the ratio test, the limit you get is

$$\lim_{n\to\infty} \frac{(n+1)!n^n}{(n+1)^{n+1}n!} = \lim_{n\to\infty} \frac{(n+1)n^n}{(n+1)^{n+1}} = \lim_{n\to\infty} \frac{(n+1)n^n}{(n+1)^n(n+1)} = \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^n.$$

To find this limit, use the technique from class: let $b_n = \left(\frac{n}{n+1}\right)^n$, so that $\ln b_n = n \ln \frac{n}{n+1}$. Rewrite this as

$$\ln b_n = \frac{\ln \frac{n}{n+1}}{1/n},$$

so that now $\lim_{n\to\infty} \ln b_n$ has the form $0/0$, so we can apply L'Hopital's rule. After doing that, we get

$$\lim_{n\to\infty} \ln b_n = \lim_{n\to\infty} \frac{\frac{n+1}{n} \cdot 1/(n+1)^2}{-1/n^2} = \lim_{n\to\infty} \frac{\frac{n+1}{n} \cdot -n^2/(n+1)^2}{1} = \lim_{n\to\infty} -\frac{n}{n+1} = -1.$$

So we've shown that $\lim_{n\to\infty} \ln b_n = -1$, and so $\lim_{n\to\infty} b_n = e^{-1} = 1/e$. Since this is less than one, the original series converges by the ratio test.

d) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n - \ln n}$

Here it looks like the $\ln n$ will be insignificant relative to the n in the denominator, and so it's tempting to compare to the divergent p-series $\sum_{n=2}^{\infty} 1/n$. But direct comparison doesn't work, since the inequality goes the wrong way. Use limit comparison with the series from the previous sentence:

$$\lim_{n\to\infty} \frac{n}{n - \ln n} = \lim_{n\to\infty} \frac{1}{1 - 1/n},$$

by L'Hopital's rule, and so the limit comes out to 1. Thus the original series behaves the same as $\sum_{n=2}^{\infty} 1/n$, and so diverges.

8. Determine whether this series converges absolutely, converges conditionally, or diverges:

$$\sum_{n=2}^{\infty} \frac{(-1)^n}{n \ln n}.$$

The series converges by the alternating series test: it's easy to check that for $n \geq 2$, $1/n \ln n$ is decreasing and tends to 0 as n goes to infinity. However, when we take absolute values, we get the series $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$. Use the integral test to show that this series diverges.

9. Determine the interval of convergence of the power series $\displaystyle\sum_{n=1}^{\infty} \frac{(x+1)^n}{n^2 3^n}$.

Using the ratio test, you should get that the series converges when $-3 < x+1 < 3$, or in other words when $-4 < x < 2$. The problem At the endpoint $x = 2$, we get the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$, which is a convergent p-series ($p = 2$). At the endpoint $x = 4$, we get the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$, which converges by the alternating series test (or converges absolutely by the $x = 2$ case, and so converges). So the interval of convergence is $[-4.2]$.