# Math 121 additional review problems for the final exam: comments and solutions

A1. The integral $\int_0^1 e^{-x^3}\,dx$ cannot be evaluated exactly. What are two ways we have seen in this course to approximate this integral? What are the advantages and disadvantages of each one?

The first way is to use Simpson's rule. When we use it with six rectangles, we get an approximation of

$$\frac{1}{18}\left(e^0 + 4e^{(-1/6)^3} + 2e^{(-1/3)^3} + 4e^{(-1/2)^3} + 2e^{(-2/3)^3} + 4e^{(-5/6)^3} + e^{-1}\right) \approx 0.80757.$$

But what's the error associated to this approximation? To use the formula, note that $a = 0$, $b = 1$, and $N = 6$. To find $K_4$, we have to calculate the fourth derivative of $e^{-x^3}$. This is a pain. If you do it, you should get

$$9e^{-x^3}(9x^8 - 36x^5 + 20x^2).$$

To find $K_4$, we need to choose it to be larger than the max absolute value that the fourth derivative attains on the interval $[0, 1]$. Unfortunately, the function is *not* strictly increasing or strictly decreasing on that interval. So we can't just take one of the endpoints. We could graph the function and select a $K_4$ that works (even if it's not the best one). That's likely the best strategy, and doing so shows you can take $K_4 = 35$. So the error we get in our approximation is at most

$$\frac{K_4(b-a)^5}{180N^4} = \frac{35}{180(6^4)} \leq 0.00016.$$

That's not bad. But it wasn't so easy to find.

The second method is to use MacLaurin series. Substituting $-x^3$ into the MacLaurin series for $e^x$, we get

$$e^{-x^3} = 1 - x^3 + \frac{x^6}{2!} - \frac{x^9}{3!} + \frac{x^{12}}{4!} - \cdots,$$

and this equality holds for all $x$. Therefore

$$\int e^{-x^3}\,dx = x - \frac{x^4}{4} + \frac{x^7}{7\cdot 2!} - \frac{x^10}{10\cdot 3!} + \frac{x^{13}}{13\cdot 4!} - \cdots,$$

And so

$$\int_0^1 e^{-x^3}\,dx = \left(1 - \frac{1}{4} + \frac{1}{7\cdot 2!} - \frac{1}{10\cdot 3!} + \frac{1}{13\cdot 4!} - \cdots\right) - 0$$
$$= 1 - \frac{1}{4} + \frac{1}{14} - \frac{1}{60} + \frac{1}{312} - \cdots$$

So we've expressed the definite integral we want as a series. Even better, the series is alternating, so we can use the alternating series error estimate. This tells us that

$$\int_0^1 e^{-x^3}\,dx = 1 - \frac{1}{4} + \frac{1}{14} - \frac{1}{60} = 0.80476\cdots,$$

with an error of at most $1/312 \approx 0.0032$. Not as good as before, you may say, but it sure was simpler. And we can make it as good as before by adding in a few more terms:

$$\int_0^1 e^{-x^3}\,dx = 1 - \frac{1}{4} + \frac{1}{7\cdot 2!} - \frac{1}{10\cdot 3!} + \frac{1}{13\cdot 4!} - \frac{1}{16\cdot 5!} + \frac{1}{19\cdot 6!} - \cdots,$$

and so

$$\int_0^1 e^{-x^3}\,dx = 1 - \frac{1}{4} + \frac{1}{14} - \frac{1}{60} + \frac{1}{312} - \frac{1}{1920} = 0.807446\cdots,$$

with an error of at most $1/(19\cdot 6!)$, which is less than 0.000075, making it better than our Simpson's rule approximation. This only involved adding up 6 numbers, and involved no complicated derivative calculations, or estimations of functions.

To compare and contrast these two methods, I'd say that they're both good, since finding the estimate for the integral is remarkably easy computationally. However, finding the error bound (specifically $K_4$ for a Simpson's rule approximation is much more difficult than any part of the series computation. So for that alone I think the series method comes out ahead. It's worth pointing out that once you find $K_4$, it becomes easy to calculate the error bound for the Simpson's rule approximation for any number of rectangles.

A2. Use material from the course to approximate each of the following numbers with a maximum error of 0.1: $e$, $\sqrt{e}$, $\sin 2$, $\ln 2$, $\sqrt{2}$. Some ideas you may want to use are Taylor Series, the alternating series error estimate, Taylor polynomials, and the Taylor polynomial error estimate.

Let's start by approximating $e$. We know that

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots,$$

and so if we truncate at $n = 4$, we get that the degree-4 MacLaurin polynomial for $f(x) = e^x$ is

$$1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24}.$$

So this tells us that

$$e = e^1 \approx 1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} \approx 2.70833.$$

How good is this approximation? Well, we can use the Taylor series error estimate. Since $n = 4$ here, we need to take $K$ to be a number such that $|f^{(5)}(x)| < K$ for $x$

in $[0,1]$. But $f^{(5)}(x) = e^x$, and $|e^x| \le e$ for $x$ in $[0,1]$, and $e < 3$. So we take $K = 3$. The error estimate is

$$\frac{K|b-a|^{n+1}}{(n+1)!} = \frac{3 \cdot 1}{5!} = \frac{3}{120} = 0.025.$$

So we've found $e$ to within the desired degree of accuracy. If you want to play more with this, find the error in the approximation to $e$ that you get by plugging 1 into the degree-10 Taylor polynomial for $e^x$. (Note that you don't need to find the actual approximation in order to find the error, though of course you're welcome to find the approximation). Then do the same for the degree-100 Taylor polynomial.

A similar use of the MacLaurin series with $x = 1/2$ works to approximate $\sqrt{e}$ (and you can get away with a Taylor polynomial of smaller degree).

Use MacLaurin series for $\sin x$ and $\ln(1+x)$ to approximate $\sin 2$ and $\ln 2$. Both of these are alternating, so you can use the alternating series error estimate rather than the Taylor polynomial error estimate.

For $\sqrt{2}$, we don't have a ready MacLaurin series for $\sqrt{x}$ in our library. So you'll need to find a Taylor polynomial manually for $\sqrt{x}$, and then calculate the error this gives when you plug in $x = 2$. Make sure you have enough terms in your Taylor polynomial so that the error is small enough.

A3. Determine a power series representation for $\tan^{-1}(x)$, find its interval of convergence, and use it to come up with an incredible power series representation for $\pi$. [Hint: start by finding a power series representation for $1/(1+x^2)$, and then integrate. When you're done, evaluate the resulting power series at $x = 1$ to get an expression involving $\pi$.]

One strategy is to find the MacLaurin series for $\tan^{-1}(x)$ directly, by computing each of its derivatives at 0, looking for a pattern, and plugging this into the formula for MacLaurin series. This is pretty labor-intensive, though.

A better way is to first find a power series representation for $1/(1+x^2)$ and then integrate. We have

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots = \sum_{n=0}^{\infty} x^n \qquad \text{for } |x| < 1,$$

and so

$$\frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = 1 - x^2 + x^4 - x^6 + \cdots = \sum_{n=0}^{\infty} (-1)^{n+1} x^{2n} \qquad \text{for } |-x^2| < 1.$$

Now $|-x^2| < 1$ is the same as $|x^2| < 1$, which is the same as $|x|^2 < 1$, which means $|x| < \sqrt{1} = 1$. So this power series representation for $1/(1+x^2)$ holds for $|x| < 1$. Now we integrate both sides:

$$\tan^{-1}(x) = C + x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots = \sum_{n=0}^{\infty} (-1)^{n+1} \frac{x^{2n+1}}{2n+1},$$

and we find that $C = 0$by setting $x = 0$. Now we know the radius of convergence of this last power series is 1, and so the series converges for $|x| < 1$. But it may also converge at the endpoints of this interval, i.e. for $x = 1$ and $x = -1$. We're most interested in $x = 1$ for this problem, so let's look at that. When $x = 1$ the series is

$$\sum_{n=0}^{\infty}(-1)^{n+1}\frac{1}{2n+1} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots .$$

Applying the alternating series test (with $a_n = 1/(2n+1)$) shows that this series converges! This may be surprising, since the power series we had before integrating did not converge for $x = 1$. But it's a nice surprise, since it shows that

$$\tan^{-1}(1) = \sum_{n=0}^{\infty}(-1)^{n+1}\frac{1}{2n+1} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots .$$

But $\tan^{-1}(1) = \pi/4$, and so we get the incredible formula

$$\pi = 4 - \frac{4}{3} + \frac{4}{5} - \frac{4}{7} + \cdots$$

A4. Use Power series to approximate

$$\int_0^1 \frac{\sin(x^2)}{x}\, dx$$

to within 0.01.

This one is very similar to the MacLaurin series method we used in A1. The key here is to note that by substituting into the MacLaurin series for $\sin x$, we get

$$\sin(x^2) = x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \cdots$$

for all $x$, and so

$$\frac{\sin(x^2)}{x} = x - \frac{x^5}{3!} + \frac{x^9}{5!} - \cdots$$

Now integrate, evaluate, and use the alternating series error estimate. It turns out you only need to sum two terms in order to get the desired accuracy!

A5. Suppose that you are at the beach, looking out over the horizon, and your eye level is 1.5 meters above the ground. Use MacLaurin series to approximate the distance that you can see to the horizon. What happens if you're looking out a window that's 18 meters high?

We discussed this one in the review session. For a solution you should consult p. 492 of the text, and use the values of $h$ from this problem.