Math 121 additional review problems for midterm 2

A1. Let $\sum_{n=1}^{\infty} a_n$ be a series. What do we mean by the sequence of terms of this series? What do we mean by the sequence of partial sums?

A2. Give the definition of convergence for a series $\sum_{n=1}^{\infty} a_n$.

A3. State the nth term test for divergence of a series $\sum_{n=1}^{\infty} a_n$.

A4. True or false. Briefly explain your answer.

(a) Let $T_2(x)$ and $T_4(x)$ be the degree 2 and 4 MacLaurin polynomials for a function $f(x)$. Then $T_4(2)$ is always a better approximation to $f(2)$ than $T_2(2)$.

(b) Let $T_{10}(x)$ be the nth Taylor polynomial for a function $f(x)$, centered at 5. If the eleventh derivative $f^{(11)}(x)$ has very large absolute value on the interval $[5, 7]$, then the error bound for the approximation of $f(7)$ by $T_{10}(7)$ will be large.

(c) A good translation of the sentence "A particle slows at a rate proportional to the square of the time it has been moving" into math language is

$$\frac{dV}{dt} = t^2,$$

where V is the velocity of the particle at time t.

(d) A sequence can only converge when its terms approach zero as n approaches infinity.

(e) The sum of the series $\sum_{n=2}^{\infty} n\left(\frac{1}{3}\right)^n$ is

$$\frac{2/9}{1-(1/3)} = \frac{2/9}{2/3} = \frac{1}{3},$$

since $r = 1/3$ and the first term is $2/9$.

(f) The series $\sum_{n=0}^{\infty} \tan^{-1}(n)$ converges.

A5. You have a cranky old jalopy whose maximum speed is 60 mph. When you floor it, its acceleration is proportional to the difference between its maximum speed and its current speed. If this machine needs only one minute to accelerate from rest to a blazing 20 mph, how long will it take to reach 50 mph? (You should start this problem by writing down a differential equation, and then solving it).

A6. Review your homework problems about Chloe! Below are the solutions to that set of homework problems.

(a) Chloe the canoeist is trying to become the first person to circumnavigate the globe in a canoe. She is making good progress, traveling at 2 m/s. Unfortunately, her paddle suddenly breaks, and she somehow forgot to bring a spare. From then on she is drifting, and she slows down at a rate proportional to the square root of her velocity. After only one second, she has already slowed to 1 m/s, and she is despairing. Will she in fact stop, and if so, how long will it take? Perhaps irrationally, she clings to a faint hope that she will still make it all the way around the world, even though the circumference of the earth is about $3 \cdot 10^7$ meters. Will she in fact make it? If so, how long does it take?

Solution: The DE here is

$$\frac{dv}{dt} = k\sqrt{v}.$$

Separating the variables gives $v^{-1/2}\ dv = k\ dt$, and then we integrate both sides to get $2\sqrt{v} = kt + C$. Dividing by 2 and absorbing that into the constants k and C, then squaring gives us

$$v(t) = (kt + C)^2.$$

Now we know that $v(0) = 2$ and $v(1) = 1$. Using these we calculate $C = \sqrt{2}$ and $k = 1 - \sqrt{2}$, and this determines $v(t)$. To find out when Chloe will stop, we look for a solution to $v(t) = 0$. This happens when $kt + C = 0$, or $t = -C/k \approx 3.414$ seconds.

To determine how far she travels before stopping, we recall that distance traveled is the integral of velocity over the given time range. So her total distance traveled is

$$\int_0^{3.414} (kt + C)^2.$$

Using the substitution $u = kt + C$ (which means $du = k\ dt$, so $dt = (1/k)\ du$), this is

$$\frac{1}{k}\left[\frac{1}{3}(kt + C)^3\right]_0^{3.414} = \frac{1}{3k}((k(3.414) + C)^3 - C^3).$$

But $(k(3.414) + C) = 0$, so this comes out to simply $-C^3/3k$, which is approximately 2.276 meters. So she falls way short of making it around the world.

(b) Chloe is back, and this time she has improved her canoe so that it drifts at a rate proportional to its velocity. Unfortunately, she got so wrapped up in the improvements to her canoe that she *again* forgot to bring a spare paddle. And wouldn't you know it, but her paddle breaks right when she's traveling at 2 m/s. After one second, she has again slowed to 1 m/s. Help her answer the same questions as in part (a).

Solution: The DE here is

$$\frac{dv}{dt} = kv.$$

Separating the variables gives $v^{-1}\ dv = k\ dt$, and then we integrate both sides to get $\ln v = kt + C$. Exponentiating both sides and simplifying the right-hand side gives us

$$v(t) = Ce^{kt}.$$

Now we know that $v(0) = 2$ and $v(1) = 1$. Using these we calculate $C = 2$ and $k = \ln(1/2) = -\ln(2)$, and this determines $v(t)$. To find out when Chloe will stop, we look for a solution to $v(t) = 0$. But e^{kt} is never zero for any value of t, so there is no solution (if you try to find a solution, you'll wind up taking ln of a negative number, which is undefined!). Therefore Chloe never stops completely.

But that doesn't mean that she will travel arbitrarily far. To determine her total distance traveled, we recall that distance traveled is the integral of velocity over the given time range. So her total distance traveled is

$$\int_0^\infty Ce^{kt}.$$

Using the substitution $u = kt$ (which means $du = k\ dt$, so $dt = (1/k)\ du$), this is

$$C \lim_{R\to\infty} \frac{1}{k} \left[e^{kt}\right]_0^R = \frac{C}{k} \lim_{R\to\infty} e^{kR} - 1.$$

Now remember that k is negative, so $\lim_{R\to\infty} e^{kR} = 0$. Thus her total distance traveled is $-C/k$, or approximately 2.885 meters. Again, she falls way short of making it around the world, even as we let time go to infinity (and even though she never completely stops!)

(c) Chloe the indomitable canoeist has by this time attracted significant media attention, and along with that came a sponsorship from a small custom canoe manufacturer. Her new canoe drifts at a rate proportional to its velocity raised to the 3/2 power. However, she has no paddle sponsor, and somehow once again forgets to bring her own spare paddle. You can guess the rest – it unfolds just as before. Help her answer the same questions as in part (a).

Solution: The DE here is

$$\frac{dv}{dt} = kv^{3/2}.$$

Separating the variables gives $v^{-3/2}\ dv = k\ dt$, and then we integrate both sides to get $-2v^{-1/2} = kt + C$. Dividing by -2, absorbing the $-1/2$ into the constants k and C, and then raising both sides to the -2 power gives

$$v(t) = \frac{1}{(kt + C)^2}.$$

Now we know that $v(0) = 2$ and $v(1) = 1$. Using these we calculate $C = 1/\sqrt{2}$ and $k + C = 1$, or $k = (\sqrt{2} - 1)/\sqrt{2}$ (note that this time k is positive!), and this determines $v(t)$. To find out when Chloe will stop, we look for a solution to $v(t) = 0$. But the reciprocal of a number can never equal zero, so there is no solution. Therefore Chloe never stops completely.

But that doesn't mean that she will travel arbitrarily far. To determine her total distance traveled, we recall that distance traveled is the integral of velocity over the given time range. So her total distance traveled is

$$\int_0^\infty (kt + C)^{-2}.$$

Using the substitution $u = kt$ (which means $du = k\ dt$, so $dt = (1/k)\ du$), this is

$$\lim_{R\to\infty} -\frac{1}{k}\left[(kt+C)^{-1}\right]_0^R = -\frac{1}{k}\lim_{R\to\infty}\frac{1}{kR+C} - \frac{1}{C} = 0 + \frac{1}{kC}$$

Thus her total distance traveled is $1/kC$, or approximately 4.828 meters. Again, she falls way short of making it around the world, even as we let time go to infinity (and even though she never completely stops!)

(d) This time, Chloe is sponsored by CanoeKing$^{\text{TM}}$, the world's largest high-tech canoe manufacturer, and this sponsorship comes with a package of a dozen spare paddles. Her sleek new canoe drifts at a rate proportional to the cube of its velocity. Alas, all her spare paddles are eaten by a band of ravenous sloths (their exponentially growing population has led to a severe famine). The same depressingly familiar sequence of events unfolds as in part (a). What happens this time?

Solution: The DE here is

$$\frac{dv}{dt} = kv^3.$$

Separating the variables gives $v^{-3}\ dv = k\ dt$, and then we integrate both sides to get $-\frac{1}{2}v^{-2} = kt + C$. Multiplying by -2, absorbing the -2 into the constants k and C, and then raising both sides to the $-1/2$ power gives

$$v(t) = \frac{1}{\sqrt{kt+C}}.$$

Now we know that $v(0) = 2$ and $v(1) = 1$. Using these we calculate $C = 1/4$ and $k+C = 1$, or $k = 3/4$ (so again k is positive!), and this determines $v(t)$. To find out when Chloe will stop, we look for a solution to $v(t) = 0$. Again, the reciprocal of a number can never equal zero, so there is no solution. Therefore Chloe never stops completely.

But that doesn't mean that she will travel arbitrarily far. To determine her total distance traveled, we recall that distance traveled is the integral of velocity over the given time range. So her total distance traveled is

$$\int_0^\infty (kt+C)^{-1/2}.$$

Using the substitution $u = kt$ (which means $du = k\ dt$, so $dt = (1/k)\ du$), this is

$$\lim_{R\to\infty}\frac{1}{k}\left[2(kt+C)^{1/2}\right]_0^R = \frac{2}{k}\lim_{R\to\infty}\sqrt{kR+C} - \sqrt{C}.$$

But this time the limit is infinite, and so given enough time, Chloe will cover any distance. So she makes it around the world! To figure out how long it will take, we look for the R such that

$$\frac{2}{k}(\sqrt{kR+C} - \sqrt{C}) = 3\cdot 10^7.$$

This gives

$$\sqrt{kR+C} = \frac{3k}{2}\cdot 10^7 + \sqrt{C} \approx \frac{9}{8}\cdot 10^7.$$

So

$$kR \approx (\frac{9}{8} \cdot 10^7)^2,$$

and this gives us that R is about $1.69 \cdot 10^{14}$ seconds, or about 5.35 million years. So Chloe had better bring a snack!