Math 232 **Fall 2008** **Quiz #3**

Read each problem carefully. Show your work.

Name SOLUTIONS

(1) Consider the following subset of P_2.

$$V = \left\{ p(t) : \int_{-1}^{1} p(t)\,dt = 0 \right\}.$$

Show that V is a subspace of P_2 and find a basis for V.

(1) Since $\int_{-1}^{1} 0\,dt = 0$, V contains the neutral (zero) element.

Suppose p, q in V. Then

$$\int_{-1}^{1} (p+q)(t)\,dt = \int_{-1}^{1} p(t) + q(t)\,dt$$

$$= \int_{-1}^{1} p(t)\,dt + \int_{-1}^{1} q(t)\,dt = 0 + 0 = 0, \text{ so}$$

$p+q$ in V.

Suppose p in V and α in $\mathbb{R}$. Then

$$\int_{-1}^{1} (\alpha p)(t)\,dt = \alpha \int_{-1}^{1} p(t)\,dt = \alpha \cdot 0 = 0, \text{ so } \alpha p \text{ in } V$$

Thus V is a subspace.

Let $p(t) = a + bt + ct^2$. Then if $0 = \int_{-1}^{1} a + bt + ct^2\,dt$

$= 2a + \frac{2c}{3}$. Thus $a = -c/3$ so

$p(t) = -c/3 + bt + ct^2 = c(t^2 - 1/3) + b(t)$.

Thus we see that $V = \mathrm{Span}(t, t^2 - 1/3)$ and since $t^2 - 1/3$ is not a multiple of t, the two functions are linearly independent.

and $(t, t^2 - 1/3)$ is a basis for V.

(2) Let V be the set of arithmetic sequences, that is, sequences of the form

$$(a, a+k, a+2k, a+3k, \ldots), \text{ for some constants } a \text{ and } k.$$

(a) Find a basis for V.

(b) Let $W = \text{Span}(\cos t, \sin t)$. Are V and W isomorphic? If so, exhibit an isomorphism. If not, explain why.

(a) $\mathcal{B} = \big((1,1,1,1,1,\ldots), (0,1,2,3,4,\ldots)\big)$

(b) V and W are isomorphic. (They are both 2-dimensional spaces.)

Here is an isomorphism:

Let T from V to W be defined by,

$$T\big((a, a+k, a+2k, a+3k, \ldots)\big) = a\cos(t) + k\sin(t).$$

(The inverse is T^{-1} from W to V defined by

$$T^{-1}\big(a\cos(t) + b\sin(t)\big) = (a, a+b, a+2b, a+3b, \ldots).)$$

(3) Let V be the space of upper triangular 2×2 matrices. Consider the linear transformation from V to V defined by

$$T\left(\begin{bmatrix} a & b \\ 0 & c \end{bmatrix}\right) = aI_2 + bP + cP^2,$$

where $P = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$.

(a) Find the matrix of T with respect to the basis

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}.$$

(b) Show that T is not an isomorphism.

(c) Find a basis for the kernel and image of T. Find the nullity and rank of T.

(a) $T\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right) = (1)I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 1\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + 0\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + 1\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

$T\left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right) = (1)P = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} = 1\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + 2\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + 3\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

$T\left(\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right) = (1)P^2 = \begin{bmatrix} 1 & 8 \\ 0 & 9 \end{bmatrix} = 1\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + 8\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + 9\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$, so

$$B = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 8 \\ 1 & 3 & 9 \end{bmatrix}.$$

(b) Since $\det(B) = 1(18-24) + 1(8-2) = -6+6 = 0$, B is not invertible. Hence T is not invertible and thus T is not an isomorphism

(c) $B \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 4 \\ 0 & 0 & 0 \end{bmatrix}$ so $\ker(B) = \operatorname{Span}\left(\begin{bmatrix} 3 \\ -4 \\ 1 \end{bmatrix}\right)$

A basis for $\ker(T)$ is $\left(3\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} - 4\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + 1\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right)$

$= \left(\begin{bmatrix} 3 & -4 \\ 0 & 1 \end{bmatrix}\right)$. And nullity $= 1$.

A basis for $\operatorname{Im}(B)$ is $\left(\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}\right)$ and thus

a basis for $\operatorname{Im}(T)$ is $\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}\right)$, and rank $= 2$.