Math 232 Linear Algebra **Fall 2008** **Test #1**

(100 points total) *Read each problem carefully. Show all work. Write your answers clearly in the Green Book. Good luck!*

(1 pt) Name SOLUTIONS

I. (6 pts) Suppose T is a linear transformation from $\mathbb{R}^4$ to $\mathbb{R}^2$ with

$$T\left(\begin{bmatrix}1\\0\\0\\0\end{bmatrix}\right)=\begin{bmatrix}4\\-2\end{bmatrix} \text{ and } T\left(\begin{bmatrix}0\\0\\1\\0\end{bmatrix}\right)=\begin{bmatrix}-1\\1\end{bmatrix}.$$

What is $T\left(\begin{bmatrix}-3\\0\\2\\0\end{bmatrix}\right)$?

$$T\left(\begin{bmatrix}-3\\0\\3\end{bmatrix}\right) = T\left(-3\vec{e_1}+2\vec{e_3}\right)$$

$$= (-3)T(\vec{e_1}) + 2T(\vec{e_3}) = (-3)\begin{bmatrix}4\\-2\end{bmatrix} + 2\begin{bmatrix}-1\\1\end{bmatrix}$$

$$= \begin{bmatrix}-14\\8\end{bmatrix}.$$

II. (8 points) Let

$$A=\begin{bmatrix}1 & 2 & 3\end{bmatrix},\ B=\begin{bmatrix}4\\5\\6\end{bmatrix},\ \text{and } \vec{x}=\begin{bmatrix}-1\\0\\2\end{bmatrix}.$$

Find each of the following *if* they exist: AB, BA, $A\vec{x}$, $B\vec{x}$.

$$AB = \begin{bmatrix}32\end{bmatrix}$$

$$BA = \begin{bmatrix}4 & 8 & 12\\5 & 10 & 15\\6 & 12 & 18\end{bmatrix}$$

$$A\vec{x} = \begin{bmatrix}5\end{bmatrix}$$

III. (15 points) Let

$$A = \begin{bmatrix} 1 & 3 & -1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & -1 \end{bmatrix}.$$

(i) Find bases for the kernel and image of A.

(ii) The image of A is an a-dimensional subspace of $\mathbb{R}^b$. The kernel of A is a c-dimensional subspace of $\mathbb{R}^d$. What are a, b, c, and d?

$$\text{rref}(A) = \begin{bmatrix} 1 & 0 & 2 & -1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \Rightarrow \begin{matrix} x_1 = -2x_3 + x_4 \\ x_2 = x_3 - x_4 \end{matrix}$$

So $\ker(A) = \left\{ \begin{bmatrix} -2s+t \\ s-t \\ s \\ t \end{bmatrix} : s,t \text{ in } \mathbb{R} \right\} = \text{Span}\left(\begin{bmatrix} -2 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix} \right)$

A basis for $\ker(T) = \left\{ \begin{bmatrix} -2 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix} \right\}$

The linearly independent columns of A are the first two, $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix}$, which give a basis for $\text{Im}(A)$. $a=2, b=3, c=2, d=4$.

IV. (10 points) Angel likes her coffee in the morning. She notices that if she drinks coffee when she wakes up there's an 80% chance she'll have coffee the next day. But if she doesn't have coffee when she wakes up there's only a 60% chance she will have coffee the next day. Set this up as a Markov chain. Find and interpret the stationary vector.

$$P = \begin{array}{c c} & \begin{matrix} C & NC \end{matrix} \\ \begin{matrix} C \\ NC \end{matrix} & \begin{bmatrix} .8 & .6 \\ .2 & .4 \end{bmatrix} \end{array}$$

$$P - I = \begin{bmatrix} -.2 & .6 \\ .2 & -.6 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & -3 \\ 0 & 0 \end{bmatrix}.$$

The kernel of $P-I$ are all multiples of $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$.

The stationary vector is $\frac{1}{4}\begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 3/4 \\ 1/4 \end{bmatrix}$.

In the long-term, Angel's chance of drinking coffee in the morning is 75%.

VII. (10 pts) Consider the linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^2$ defined by

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x+y \\ 2x-2y \end{bmatrix}.$$

Determine if the transformation is invertible. If it is, find the inverse transformation $T^{-1}\left(\begin{bmatrix} x \\ y \end{bmatrix}\right)$. It's easiest to work with matrices.

Here $T(\vec{x}) = A\vec{x}$, where $A = \begin{bmatrix} 1 & 1 \\ 2 & -2 \end{bmatrix}$. The inverse of A is

$$\left[\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 2 & -2 & 0 & 1 \end{array}\right] \xrightarrow{\text{rref}} \left[\begin{array}{cc|cc} 1 & 0 & 1/2 & 1/4 \\ 0 & 1 & 1/2 & -1/4 \end{array}\right]. \quad A^{-1} = \begin{bmatrix} 1/2 & 1/4 \\ 1/2 & -1/4 \end{bmatrix}.$$

$$\text{So } T^{-1}\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = A^{-1}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1/2\,x + 1/4\,y \\ 1/2\,x - 1/4\,y \end{bmatrix}.$$

VIII. (10 pts) Consider the vectors $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$, and $\begin{bmatrix} -2 \\ 1 \\ 4 \end{bmatrix}$. Determine whether or not they are linearly independent. If not, find a redundant vector and write it as a linear combination of the other vectors.

Suppose $\alpha\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \beta\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} + \gamma\begin{bmatrix} -2 \\ 1 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, or $(*)$

$\begin{bmatrix} 1 & 3 & -2 \\ 1 & 2 & 1 \\ 1 & 1 & 4 \end{bmatrix}\begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$. Row-reducing the augmented coefficient matrix

gives $\left[\begin{array}{ccc|c} 1 & 3 & -2 & 0 \\ 1 & 2 & 1 & 0 \\ 1 & 1 & 4 & 0 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 7 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$. The system has infinitely many solutions and thus there is a non-trivial solution to $(*)$. The vectors are not linearly independent. We also see that $\begin{bmatrix} -7 \\ 3 \\ 1 \end{bmatrix}$ is a solution and thus

$$(-7)\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + (3)\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} + (1)\begin{bmatrix} -2 \\ 1 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} -2 \\ 1 \\ 4 \end{bmatrix} = 7\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - 3\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}.$$

V. (10 points) Let V be the set of vectors of the form $\begin{bmatrix} a \\ b \\ 0 \\ a+b \end{bmatrix}$. Show whether or not V a subspace.

V is a subspace: (i) Letting $a=b=0$ shows $\vec{0}$ is in V.

(ii) Let $\vec{x} = \begin{bmatrix} a \\ b \\ 0 \\ a+b \end{bmatrix}$ and $\vec{y} = \begin{bmatrix} c \\ d \\ 0 \\ c+d \end{bmatrix}$. Then $\vec{x}+\vec{y} = \begin{bmatrix} a+c \\ b+d \\ 0 \\ (a+c)+(b+d) \end{bmatrix}$, which is in V

(iii) Let $\vec{x} = \begin{bmatrix} a \\ b \\ 0 \\ a+b \end{bmatrix}$ and α in $\mathbb{R}$. Then $\alpha\vec{x} = \begin{bmatrix} \alpha a \\ \alpha b \\ 0 \\ \alpha(a+b) \end{bmatrix}$, which is in V.

Thus V is a subspace.

VI. (6 pts) Give an example of a 4×5 matrix with $\dim(\ker(A)) = 3$.

By FTLA, $\operatorname{rank}(A) = 2$.

Take $A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$.

IX. (6 pts) Let A be an 8×8 matrix with $A^2 = 0$.

(a) Let $\vec{x}$ be in $\mathrm{im}(A)$. Then $\vec{x} = A\vec{z}$ for some $\vec{z}$.
And $A\vec{x} = A(A\vec{z}) = A^2\vec{z} = \vec{0}$, so $\vec{x}$ is in $\ker(A)$.
Thus $\mathrm{rank}(A) \leq \mathrm{nullity}(A)$.

(b) By FTLA, $8 = \mathrm{nullity}(A) + \mathrm{rank}(A) \geq \mathrm{rank}(A) + \mathrm{rank}(A) = 2\,\mathrm{rank}(A)$.
Thus $\mathrm{rank}(A) \leq 4$.

(a) Show that every vector in the image of A is also in the kernel of A. That is, show that $\mathrm{im}(A) \subseteq \ker(A)$.

(b) Now show that $\mathrm{rank}(A) \leq 4$.

X. (18 points—3 pts each) TRUE OR FALSE. (You do not need to give a justification.)

(i) A linear system with fewer unknowns that equations must have infinitely many solutions.

(ii) The formula $(A^2)^{-1} = (A^{-1})^2$ holds for all invertible matrices A.

(iii) If $A^2+3A+4I_n = 0$ for an $n \times n$ matrix A, then A must be invertible.

(iv) If $\vec{v_1}, \ldots, \vec{v_p}$ span $\mathbb{R}^p$, then they must form a basis for $\mathbb{R}^p$.

(v) $\mathbb{R}^3$ is a subspace of $\mathbb{R}^4$.

(vi) There exists a 5×5 matrix such that $\mathrm{im}(A) = \ker(A)$.

(i) F — Could have 0 solutions

(ii) T

(iii) T — $I_n = \frac{1}{4}(-A^2 - 3A) = A\underbrace{(-A-3I)/4}_{A^{-1}}$.

(iv) T

(v) F

(vi) False (by FTLA).