Math 232 Linear Algebra **Winter 2008** **Quiz #2**

Read each problem carefully. Show all work. Good luck!

Name Solutions

(1) Suppose $\vec{u_1}, \vec{u_2}, \vec{u_3}, \vec{u_4}$ are orthonormal vectors in $\mathbb{R}^7$. Find the length of

$$\vec{x} = 5\vec{u_1} - \vec{u_2} + 2\vec{u_3} - 3\vec{u_4}.$$

$$\|\vec{x}\|^2 = (5\vec{v}_1 - \vec{v}_2 + 2\vec{v}_3 - 3\vec{v}_4) \cdot (5\vec{v}_1 - \vec{v}_2 + 2\vec{v}_3 - 3\vec{v}_4)$$

$$= 25\|\vec{v}_1\|^2 + (-1)^2\|\vec{v}_2\|^2 + (2)^2\|\vec{v}_3\|^2 + (-3)^2\|\vec{v}_4\|^2 + 0$$

$$= 25 + 1 + 4 + 9 = 39 \quad \text{so} \quad \|\vec{x}\| = \sqrt{39}.$$

(One can also use the Pythagorean Theorem.)

(2) Compute the determinant of A and determine the value(s) of k for which A is invertible.

$$A = \begin{bmatrix} 1 & 2 & 3 & k \\ 2 & 1 & 4 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 1 & 3 & -1 \end{bmatrix}.$$

Expand along 3 row:

$$\det(A) = (-2)\det\begin{bmatrix} 1 & 3 & k \\ 2 & 4 & 0 \\ 0 & 3 & -1 \end{bmatrix}$$

$$= (-2)\left[k(6-0) - (4-6)\right] = (-2)\left[6k + 2\right]$$

$$= \underline{-12k - 4}$$

A is invertible iff $\det(A) \neq 0$ iff $k \neq -1/3$.

(3) Let V be the subspace of $\mathbb{R}^4$ with orthonormal basis

$$\left\{ \begin{bmatrix} 1/2 \\ -1/2 \\ 1/2 \\ 1/2 \end{bmatrix}, \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ -1/2 \end{bmatrix} \right\}. = \{\vec{v}_1, \vec{v}_2\}$$

Find the projection of $2\vec{e}_4$ onto V.

$$\text{proj}_V(2\vec{e}_4) = (2\vec{e}_4 \cdot \vec{v}_1)\vec{v}_1 + (2\vec{e}_4 \cdot \vec{v}_2)\vec{v}_2$$

$$= (1)\begin{bmatrix} 1/2 \\ -1/2 \\ 1/2 \\ 1/2 \end{bmatrix} + (-1)\begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ -1/2 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}.$$

(4) Let T be an orthogonal transformation from $\mathbb{R}^3$ to $\mathbb{R}^3$. Suppose

$$\vec{x} = T\left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right).$$

Which vector is a possible choice for $\vec{x}$?

(i) $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ (ii) $\begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix}$ (iii) $\begin{bmatrix} 1 \\ 0 \\ \sqrt{2} \end{bmatrix}$ (iv) $\begin{bmatrix} \sqrt{3} \\ \sqrt{3} \\ \sqrt{3} \end{bmatrix}$

Since

$$\left\| \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\| = \sqrt{3} \text{ and } \left\| T\left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right) \right\| = \sqrt{3}$$

$$= \left\| \begin{bmatrix} 1 \\ 0 \\ \sqrt{2} \end{bmatrix} \right\|.$$

(Orthogonal transformations preserve length

(5) How are SAT scores affected by family income? Data were collected from a sample of five students. (The reported annual income is in tens of thousands of dollars.)

ID	Family income	SAT score
1	6	560
2	6	545
3	8	590
4	10	615
5	7	600

The data were fit to a quadratic polynomial $y = a + bx + cx^2$ using the least squares algorithm.

(i) What is the A matrix for this problem?

(ii) The least squares solution is obtained by projecting a certain vector $\vec{b}$ onto a certain subspace. What is $\vec{b}$? What is the name of that subspace and what is its dimension? (Obtain the dimension by inspection only—do not do any computations.)

(iii) Suppose the least squares solution is

$$\vec{x}^* = \begin{bmatrix} 200 \\ 90 \\ -4 \end{bmatrix}.$$

What does the model predict for a student's SAT score if their annual family income is \$100,000?

(i) $A = \begin{bmatrix} 1 & 6 & 36 \\ 1 & 6 & 36 \\ 1 & 8 & 64 \\ 1 & 10 & 100 \\ 1 & 7 & 49 \end{bmatrix}$

(ii) $\vec{b} = \begin{bmatrix} 500 \\ 545 \\ 590 \\ 615 \\ 600 \end{bmatrix}$; The subspace is Image(A), which has dimension = 3.

(iii) $SAT = 200 + 90(10) - 4(10)^2$

$= 200 + 900 - 400 = \underline{\underline{700}}$.