Math 141 **Winter 2008** **Midterm Exam**

(80 points total) *Read each problem carefully. Show all work. Make sure to indicate your solution. Good luck!*

SOLUTIONS

(1) (10 pts) Find the fixed points for the following recurrence relation and test for stability.

$$x(n) = (x(n-1))^2 + x(n-1) - \frac{1}{4}, \quad n \geq 1.$$

Solve $\bar{x} = \bar{x}^2 + \bar{x} - 1/4$ or $\bar{x}^2 = 1/4$. So $\bar{x} = 1/2$ and $-1/2$ for the fixed points. For stability, take the derivative of $f(x) = x^2 + x - 1/4$, so $f'(x) = 2x + 1$. Now $|f'(1/2)| = 2 > 1$ and $|f'(-1/2)| = 0 < 1$ so the fixed point 1/2 is not stable and the fixed point $-1/2$ is.

(2) (10 pts) A linear, third-order, constant coefficient recurrence relation for $x(n)$ has eigenvalues 1, 1/2, and $-1/4$. Describe in words what happens to $x(n)$ as n gets large.

The closed form solution will be of the form $x(n) = c_1 1^n + c_2(1/2)^n + c_3(-1/4)^n \to c_1$, as $n \to \infty$. Thus, for large n, $x(n)$ will converge to a constant.

(3) (15 pts) A population has a 7% birth rate and a 4% death rate. In addition, every year 10% of the population emigrates, and 3,500 people immigrate.

(i) Use a compartment analysis sketch to describe this population.

(ii) If the population starts at 10,000, what happens after many years?

Let $x(n)$ be the population at time n. Then we have that $x(n) - x(n-1) = .07x(n-1) - .04x(n-1) + .1x(n-1) + 3,500$ or $x(n) = .93x(n-1) + 3,500$. The fixed point is found by solving $\bar{x} = .93\bar{x} + 3,500$,

which gives $\bar{x} = 50,000$. Thus the population increases and stabilizes at 50,000.

(4) (15 pts) In Ozfield there are three weather conditions—Snow, Sun, and Rain. It always Rains the day after it Snows. And it is always Sunny the day after it Rains. But when it's Sunny, the next day all weathers are equally likely.

(i) Exhibit the transition matrix for this Markov chain.

Letting the states be Snow, Sun, and Rain, the transition matrix is

$$T = \begin{bmatrix} 0 & 1/3 & 0 \\ 0 & 1/3 & 1 \\ 1 & 1/3 & 0 \end{bmatrix}.$$

(ii) If it's Raining on Sunday. What's the probability that it will be Sunny on Wednesday?

Letting $x(0) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, **we want** $x(3) = T^3 x(0)$. **We find that** $x(1) = Tx(0) = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $x(2) = Tx(1) = \begin{bmatrix} 1/3 \\ 1/3 \\ 1/3 \end{bmatrix}$, **and** $x(3) = Tx(2) = \begin{bmatrix} 1/9 \\ 4/9 \\ 4/9 \end{bmatrix}$. **So the probability of being Sunny is** $4/9$.

(iii) Find and interpret the steady state vector. (If you aren't able to find the steady state vector, use $q = (2/11, 5/11, 4/11)$ for the second part of this question.)

To find the steady state vector solve $Tq = q$. **Let** $q = \begin{bmatrix} a \\ b \\ 1-a-b \end{bmatrix}$ **and obtain the equations** $a = (1/3)b$, $b = 1 - a - (2/3)b$. **Solving these gives** $q = (1/2, 1/6, 1/3)$. **Thus, in Oz, about half the days are Snowing, 1/6 are Sunny and 1/3 are Raining.**

(5) (15 pts) Suppose an animal lives three years. The first year it is immature and does not reproduce. The second year it is an adolescent and reproduces at a rate of 0.8 female offspring per female indivdiual. The last year it is an adult and produces 3.5 female offspring per female individual. Further suppose that 80% of the first-year females survive to become second-year females, and 90% of the second-year females survive to become third-year females. All third-year females die.

(i) Draw a state diagram for this scenario and construct the Leslie matrix.

$$L = \begin{bmatrix} 0 & .8 & 3.5 \\ .8 & 0 & 0 \\ 0 & .9 & 0 \end{bmatrix}.$$

(ii) To find the eigenvalues and eigenvectors for this matrix, *Mathematica* produces the following output.

`Eigenvalues[mat]` $= \{1.52, -0.76 + 1.04i, -0.76 - 1.04i\}$

and

`Eigenvectors[mat]` $=$

$\{\{-0.85, -0.45, -0.27\}$,
$\{-0.80, 0.29 + 0.40i, 0.11 - 0.33i\}$,
$\{-0.80, 0.29 - 0.40i, 0.11 + 0.33i\}\}$

(iii) Use this information to describe the population growth and distribution after many years.

The dominant eigenvalue is 1.52, and thus the population eventually grows by about 52% per year. The normalized dominant eigenvector (obtained by dividing the given eigenvector by the sum of its components) is $(.54, .29, .17)$. Thus in the long run, about 54% of the animals are immature, 29% are adolescent, and the remaining 17% are adult.

(6) (15 pts) Here is the transition matrix for an absorbing Markov chain.

$$T = \begin{bmatrix} 0 & .3 & 0 & 0 & 0 \\ .7 & 0 & .3 & 0 & 0 \\ 0 & .7 & 0 & 0 & 0 \\ .3 & 0 & 0 & 1 & 0 \\ 0 & 0 & .7 & 0 & 1 \end{bmatrix}.$$

The states are labeled 1, 2, 3, 4, and 5, with states 4 and 5 absorbing. The fundamental matrix is

$$F = \begin{bmatrix} 1.4 & .5 & .2 \\ 1.2 & 1.7 & .5 \\ .8 & 1.2 & 1.4 \end{bmatrix}.$$

(i) From state 2, how many steps, on average, will it take to be absorbed? **3.4, which is the sum of the 2nd column of the fundamental matrix.**

(ii) From state 2, how many times, on average, will the process return to state 2? **1.7, the second entry in the second column of the fundamental matrix.**

(iii) From state 2, what's the probability that the process will be absorbed in one step? **0, the sum of the 4th and 5th entries of column 2 of the transition matrix.**

(iv) From state 2, what's the probability that the process will eventually be absorbed in state 4? **15.15%. Compute BF, where B is the bottom left-hand submatrix of T, and F is the fundamental matrix. The first entry of the second column of the resulting matrix gives the absorption probabilities.**

(v) From state 2, what's the probability that the process will never be absorbed? **0. In an absorbing Markov chain the process eventually becomes absorbed.**