A Conjecture of Gleason on the Foundations of Geometry

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Abstract

Felix Klein emphasized the intrinsic connection between symmetry groups and geometries in his Erlangen Program. Perhaps motivated by Klein, Gleason ([5]) posed a very general conjecture on topologizing symmetry groups that he regarded as fundamental for a general study of geometries. Gleason in fact proved his conjecture in a very special case. The purpose of this paper is to show that Gleason's general conjecture is false as originally stated and that it is true only under very strong hypotheses. Along the way new general results in descriptive set theory are proved about a class of functions that behave like but are distinct from functions of Baire class 1.

1 Introduction

Felix Klein emphasized the intrinsic connection between symmetry groups and geometries in his Erlangen Program. Perhaps motivated by Klein, Gleason ([5]) posed a very general conjecture on topologizing symmetry groups that he regarded as fundamental for a general study of geometries. Gleason in fact proved his conjecture in a very special case. The purpose of this paper is to show that Gleason's general conjecture is false as originally stated and that it is true only under very strong hypotheses. Along the way new general results in descriptive set theory are proved about a class of functions that behave like but are distinct from functions of Baire class 1.

Paraphrasing Gleason, we ask: if G is an abstract group of homeomorphisms of a topological space M, under what circumstances can G be given a topology such that the pair (G, M) is a topological transformation group? That is, when can G be given a (reasonable) topological group topology such that the mapping $(g, m) \mapsto g(m)$, $G \times M \mapsto M$, is continuous? Gleason gives a plausibility argument relating this question to very general geometries. Define a (topological) geometry as a topological space in which certain lines (subsets homeomorphic to $\mathbb{R}$) are distinguished. Let G be the group of automorphisms of a geometry M, i.e., the group of homeomorphisms of M that induce a permutation of the lines of M. It is reasonable to assume that M is homogeneous, i.e., that G acts transitively on M. It is also reasonable to assume some local uniqueness for lines which in turn implies that G is not "too big". Gleason's hope was that every topological geometry of finite dimension that satisfies some weak geometrical axioms must be the homogeneous space of some Lie group. In this context Gleason defined a frame for the action of G on M to be a an element $(m_1, \ldots, m_n) \in M^n$ such that the mapping $g \mapsto (g(m_1), \ldots, g(m_n))$, $G \mapsto M^n$, is an injection. In this paper such a frame will be called a finite frame and n will be called the size of the finite frame. It is convenient to define a countably infinite frame to be an element $(m_1, m_2, \ldots) \in M^{\mathbb{N}}$ such that the mapping $g \mapsto (g(m_1), g(m_2), \ldots)$, $G \mapsto M^{\mathbb{N}}$, is an injection. Guided by this very general geometric model, Gleason considered the following axioms:

Axiom 1.
M is a Polish space and G is a group of homeomorphisms of M that acts transitively on M;

Axiom 2(a).
There is a finite frame $(m_1, \ldots, m_n) \in M^n$ for the action of G on M such that

$$\{(g(m_1), \ldots, g(m_n), g(q)) \mid g \in G\} \subseteq M^{n+1}$$

is an analytic set for each $q \in M$.

Gleason's Conjecture.
If G and M satisfy Axiom 1 and Axiom 2(a), then G can be given a Polish group topology such that the pair (G, M) is a topological transformation group.

The assumption in Axiom 1 that M is a Polish space and that G is a group of homeomorphisms of G is a very mild condition. The assumption that G acts transitively on M is a very restrictive and powerful assumption. The existence of a finite frame for the action of G on M in Axiom 2(a) corresponds to the assumption that lines are locally unique and that G is not too big. The analyticity condition in Axiom 2(a) is a smoothness assumption and is somewhat problematic in that it does not correspond to any obvious geometric assumption. However, Gleason pointed out that some assumption like Axiom 2(a) is needed.

Specifically, if $M = \mathbb{C}^2 - \{(0,0)\}$ and G is the group of nonzero quaternion matrices, then there is even a simply transitive action of G by homeomorphisms on M that violates Axiom 2(a) and such that there is no way that G can be made into a Polish group such that (G, M) is a topological transformation group.

Gleason ([5]) proved his conjecture in the very special case of size one frames, i.e., Gleason proved his conjecture if G acts simply transitively on M. However, in this case notice that the analytic set $\{(g(m), g(q)) \mid g \in G\} \subseteq M^2$ is the graph of a function on M for each $q \in M$. Therefore the mapping $(g(m), g(q)) \mapsto g(m)$, $\{(g(m), g(q)) \mid g \in G\} \mapsto M$, is a continuous bijection and therefore a Borel isomorphism by [13], Theorem 4.2. Hence, $\{(g(m), g(q)) \mid g \in G\}$ is actually a Borel set and not merely an analytic set. On the other hand, if Gleason's conjecture is true, then the mapping $g \mapsto (g(m_1), \ldots, g(m_n), g(q)))$, $G \mapsto M^{n+1}$, is a continuous injection for each $q \in M$. Hence, Lusin-Souslin's theorem implies that $\{(g(m_1), \ldots, g(m_n), g(q)) \mid g \in G\} \subseteq M^{n+1}$ is in fact a Borel set. This suggests that Axiom 2(a) should be replaced by the stronger

Axiom 2(b).
There is a finite frame $(m_1, \ldots, m_n) \in M^n$ for the action of G on M such that

$$\{(g(m_1), \ldots, g(m_n), g(q)) \mid g \in G\} \subseteq M^{n+1}$$

is a Borel set for each $q \in M$.

However, Axiom 2(b) still will not be sufficient to guarantee that the conclusion of Gleason's Conjecture is true. This situation will be further clarified in Section 2, where it will be shown that there is a G and an M that satisfy Axiom 1 and Axiom 2(a) such that $\{(g(m_1), \ldots, g(m_n), g(q)) \mid g \in G\}$ is a K_σ for every $q \in M$, but there will be no way to make G into a Polish group let alone have (G, M) be a Polish transformation group. Notice that in this counterexample the G-orbit of any frame is also a K_σ since the continuous image of any K_σ is again a K_σ. This example suggest a further strengthening of Axiom 2(b).

Axiom 2(c).
There is a finite frame $F = (m_1, \ldots, m_n) \in M^n$ for the action of G on M such that the G-orbit of the frame

$$\{(g(m_1), \ldots, g(m_n)) \mid g \in G\} \subseteq M^n$$

is a G_δ in M^n and

$$\{(g(m_1), \ldots, g(m_n), g(q)) \mid g \in G\} \subseteq M^{n+1}$$

is a Borel subset of M^{n+1} for each $q \in M$.

Axiom 2(c) is consistent with the size one frame case proved by Gleason since in that simply transitive case the G-orbit of a frame, i.e., a single point, is M itself, trivially a G_δ subset of M. We also consider two weaker variants of this axiom. To motivate the next axiom note that there exist transitive Polish transformation groups (G, M) with no finite frame but with a countably infinite frame. For example, take M to be the unit sphere of a separable infinite dimensional complex Hilbert space $\mathcal{H}$ and let G be the full unitary group of $\mathcal{H}$. Any orthonormal basis is then a frame for the action of G on M.

Axiom 2(d).
There is a countably infinite frame $F = (m_1, m_2, \ldots) \in M^{\mathbb{N}}$ for the action of G on M such that the G-orbit of the frame

$$\{(g(m_1), g(m_2), \ldots) \mid g \in G\} \subseteq M^{\mathbb{N}}$$

is a G_δ-subset of $M^\mathbb{N}$ and

$$\{(g(m_1), g(m_2) \ldots, g(q)) \mid g \in G\} \subseteq M^\mathbb{N} \times M$$

is a Borel set for each $q \in M$.

and

Axiom 2(e).
There is a dense sequence $\{m_\ell\}_{\ell \geq 1}$ in M such that the G-orbit $\{(g(m_1), g(m_2), \ldots) \subseteq M^\mathbb{N}$ is a G_δ-subset of $M^\mathbb{N}$ and

$$\{(g(m_1), g(m_2) \ldots, g(q)) \mid g \in G\} \subseteq M^\mathbb{N} \times M$$

is a Borel set for each $q \in M$.

Of course if $\{m_\ell\}_{\ell \geq 1}$ is dense in M, then $(m_1, m_2, \ldots)$ is a countably infinite frame for G. It will be proved that Axiom 2(c) implies Axiom 2(e) which in turn implies Axiom 2(d). The purpose of this paper is to prove that the conclusion of Gleason's conjecture is indeed true if Axiom 1 and Axiom 2(c) or Axiom 2(d) or Axiom 2(e) hold.

An example is given in Section 2 plausibly showing that the G_δ condition in the axioms is needed. A new general result in descriptive set theory is given in Section 3. Gleason's results for the simply transitive case (frames of size one) are recalled and generalized for the convenience of the reader in Section 4. The relations among Axiom 2(c), Axiom 2(d) and Axiom 2(e) are proved in Section 5 together with the proof of the general conjecture that Axiom 1 and Axiom 2(e) imply that G can be made into a Polish group so that (G, M) is a topological transformation group. An application of this general result is given in Section 6.

2 A Counterexample

The purpose of this section is to show that Axiom 1 and Axiom $2(b)$ can hold even though the conclusion to Gleason's Conjecture is false.

Lemma 1 .
Let $K \subseteq \mathbb{R}$ be an uncountable compact set whose elements are linearly independent over $\mathbb{Q}$. Such a K exists. Let H be the additive subgroup of $(\mathbb{R}, +)$ algebraically generated by K. Then H is σ-compact and there is no algebraic isomorphism of H with any Polish group.

Proof:

Von Neumann [17] proved that there is a injection $f : \langle 0, +\infty \rangle \mapsto \mathbb{R}$ whose range consists of numbers that are algebraically independent over $\mathbb{Q}$. A simple inspection of von Neumann's construction shows that f is a Borel mapping. Thus the range of f is an uncountable Borel set and therefore contains a compact perfect set. Thus such a K exists. It is simple to check that H is σ-compact. As an abstract group $H = \oplus_{x \in K} \mathbb{Z}x$. Suppose that G is a Polish group and $\varphi : G \mapsto H$ is an algebraic isomorphism. Then Lemma 2 and Theorem 1 of Dudley [3] imply that φ is continuous if H is given the discrete topology. In particular $e_G = \varphi^{-1}(e_H)$ is open and therefore G is a discrete Polish group. This implies that G is countable, a contradiction, since G is algebraically isomorphic to H, an uncountable group. $\square$

Proposition 2 .
There exist a σ-compact subgroup G of a Polish group K and a Polish space M such that (G, M) is a transitive topological transformation group with a frame of every size $n \geq 2$ such that there is no algebraic isomorphism of G with any Polish group.

Proof:

Notice that under the assumptions of the proposition if there is a frame of size n for the action of G on M, then there is a frame of size $n + 1$ for the action of G on M by merely adding any element of M as the $n + 1$-st entry to the original frame. It therefore suffices to prove the proposition to show the existence of a frame of size 2.

Let A be the exponentiation of the additive subgroup H of the reals given in Lemma 1, so that A is a subgroup of the multiplicative group of positive reals. There is no algebraic isomorphism of A with any Polish group since A is algebraically and topologically isomorphic to H. Let B be the additive group of the reals and let $G = B \rtimes A$ be the natural semidirect product. G is a σ-compact subgroup of the classical $ax + b$ group, a Polish group. If M is the real numbers, then (G, M) is a transitive topological transformation group and $(1, -1) \in M^2$ is a frame for (G, M). Suppose that L is a Polish group and $\varphi : L \mapsto G$ is an algebraic isomorphism. It is simple to check that A is maximal abelian in G. $A_L = \varphi^{-1}(A)$ is maximal abelian in L and therefore is closed in L and is itself a Polish group. This is a contradiction since $\varphi|A_L : A_L \mapsto A$ is an algebraic isomorphism of A with a Polish group A_L. $\square$

Though unrelated to the other results of this section, it should be noted that the elements of a frame for a transitive group action are not at all analogous to a basis for a vector space, even after extraneous elements of the frame are omitted. This is the case even for finite groups. For example, let G be the symmetric group on a set of size six $X = \{1, 2, 3, 4, 5, 6\}$. If $\emptyset \neq S \subseteq X$, let $G_S = \{g \in G \mid g(s) = s \text{ for all } s \in S\}$ and let $M = G/G_{\{1,2,3\}}$, a transitive G-space. Choose g_1, g_2 and $g_3 \in G$ such that $g_1 G_{\{1,2,3\}} g_1^{-1} = G_{\{4,5,6\}}$, $g_2 G_{\{1,2,3\}} g_2^{-1} = G_{\{1,2,4\}}$ and $g_3 G_{\{1,2,3\}} g_3^{-1} = G_{\{1,2,5\}}$. Then $(G_{\{1,2,3\}}, g_1 G_{\{1,2,3\}})$ and $(G_{\{1,2,3\}}, g_2 G_{\{1,2,3\}}, g_3 G_{\{1,2,3\}})$ are two frames for the action of G on M of different sizes that cannot be reduced in size by omitting judiciously chosen elements.

3 A Descriptive Set Theory Result

We start with a trivial observation. If X and Y are Polish spaces, $\varphi : X \mapsto Y$ is continuous, $V \subseteq Y$ is open, $U \subseteq X$ is open, and $U \cap \varphi^{-1}(V) \neq \emptyset$, then $U \cap \varphi^{-1}(V)$ is a nonempty open subset of X and therefore is nonmeager in X. What about the converse? That is, suppose that X and Y are Polish spaces and $\varphi : X \mapsto Y$ satisfies the property that if $V \subseteq Y$ is open and if $U \subseteq X$ is open and if $U \cap \varphi^{-1}(V) \neq \emptyset$, then $U \cap \varphi^{-1}(V)$ is nonmeager in X. Does this imply that φ satisfies some sort of nontrivial continuity property? In general, the answer is no. For example, let $X = \mathbb{R}$, $Y = \{0, 1\}$, $B \subseteq \mathbb{R}$ a Bernstein set ([16], pp. 32 – 33) and let $\varphi = \chi_B$, the characteristic function of B. The construction of $B = \chi_B^{-1}(1)$ shows that $B^c = \chi_B^{-1}(0)$ is also a Bernstein set, i.e., neither B nor B^c contains a compact perfect set. If $A \subseteq \mathbb{R}$ is an uncountable analytic set, then $\varphi|A$ cannot be continuous. This follows since A contains a compact perfect set K. If $x \in K \cap B$, then every relative neighborhood of x in K contains a compact perfect set and therefore contains of point of B^c, showing that $\chi_B|K$ cannot be continuous. So some a priori weak smoothness assumption is needed on φ to in order to conclude that φ has some sort of reasonable continuity property.

Before proceeding further, we set up some notation and recall some very general results about Baire category. Let X be a topological space, $A \subseteq X$, and $M(A)$ the union of all open sets $V \subseteq X$ such that $V \cap A$ is meager in X. Then $M(A)$ is open in X and $A \cap \mathrm{cl}_X(M(A))$ is meager in X ([11], p. 201). Define $D(A) = M(A)^c$, a closed subset of X, and define $ID(A) = \mathrm{Int}(D(A))$, an open subset of X. $D(A)$ is the set of points in X at which A is not locally of the first category in X and $ID(A)$ is the interior of the set of points in X at which A is not locally of the first category in X. The following lemma consists of well known results that are left as an exercise for the reader and most of which can be extracted from Kuratowski [12].

Lemma 3 .
Let X be a topological space and A, B, A_n, $A_\iota \subseteq X$. Then:
(1) if $A \subseteq B$, then $M(B) \subseteq M(A)$ and therefore $D(A) \subseteq D(B)$;
(2) $M(A \cup B) = M(A) \cap M(B)$ and therefore $D(A \cup B) = D(A) \cup D(B)$;
(3) $\mathrm{cl}_X(A)^c \subseteq M(A)$ and therefore $D(A) \subseteq \mathrm{cl}_X(A)$;
(4) $ID(A) = X - \mathrm{cl}_X(M(A))$;
(5) if $U \subseteq X$ is open, then $D(U) - U$ is meager;
(6) $D(A) = \emptyset$ if and only if A is meager;
(7) $A - D(A)$ is meager and $D(A - D(A)) = \emptyset$;
(8) $A - ID(A)$ is meager and $D(A - ID(A)) = \emptyset$;
(9) $D(A) - D(B) \subseteq D(A - B)$;
(10) $D(\cap_\iota A_\iota) \subseteq \cap_\iota D(A_\iota)$;
(11) $\cup_\iota D(A_\iota) \subseteq D(\cup_\iota A_\iota)$;
(12) if $U \subseteq X$ is open, then $U \cap D(A) = U \cap D(U \cap A)$;
(13) $D(D(A)) = D(A)$;
(14) $D(A) = \mathrm{cl}_X(ID(A))$;
(15) $ID(A) = \mathrm{Int}(\mathrm{cl}_X(ID(A)))$;
(16) $ID(A) = \emptyset$ if and only if A is meager;
(17) $D(\cup_{n \geq 1} A_n) - \cup_{n \geq 1} D(A_n)$ is nowhere dense;
(18) if A is nonmeager, then $A \cap ID(A)$ is nonempty;
(19) if $A \subseteq U$, where U is open and A is nonmeager, then $U \cap ID(A) \neq \emptyset$;

Lemma 4 *Let X be a topological space and let $A \subseteq X$ be any set. Then $A \subseteq D(A)$ if and only if the following property holds: whenever $U \subseteq X$ is open and $U \cap A \neq \emptyset$, then $U \cap A$ is nonmeager in X.*

Proof:

Suppose $A \subseteq D(A)$ and let $U \subseteq X$ be open such that $U \cap A \neq \emptyset$. Let $x \in U \cap A$. Then $x \in D(A)$ and hence $x \notin M(A)$. Since U is an open neighborhood of x, it follows that $U \cap A$ is nonmeager.

Conversely, suppose whenever $U \subseteq X$ is open and $U \cap A \neq \emptyset$, then $U \cap A$ is nonmeager. Let $x \in A$ be arbitrary. If U is any open neighborhood of x, we have that $x \in U \cap A \neq \emptyset$, and hence $U \cap A$ is nonmeager by hypothesis. This implies $x \notin M(A)$, i.e. $x \in D(A)$. So $A \subseteq D(A)$. $\square$

Corollary 5 .
Let X and Y be topological spaces and let $\varphi : X \mapsto Y$. Then the following are equivalent:
(1) $\varphi^{-1}(V) \subseteq D(\varphi^{-1}(V))$ for every open $V \subseteq Y$;
(2) if $V \subseteq Y$ is open, $U \subseteq X$ is open and $U \cap \varphi^{-1}(V) \neq \emptyset$, then $U \cap \varphi^{-1}(V)$ is nonmeager in X.

Recall that if X be a topological space, then a set $A \subseteq X$ is said to be a set with the Baire property in X if there exists an open set $U \subseteq X$ such that $A \triangle U$ is meager in X. Let $\mathcal{BP}(X)$ be the collection subsets of X with the Baire property. $\mathcal{BP}(X)$ is the smallest σ-algebra of subsets of X generated by the open sets and the first category sets and therefore contains the Borel subsets of X. It is a nontrivial fact that if X is a Polish space then $\mathcal{BP}(X)$ contains the analytic subsets of X. Again, the following lemma consists of well known facts that are left as an exercise for the reader or can be gleaned from various places in Kuratowski [12].

Lemma 6 .
Let X be a topological space and let $A \subseteq X$. Then the following statements are equivalent:
(0) $A \in \mathcal{BP}(X)$;
(1) $A = G \cup M$, where G is a G_δ and M is meager in X;
(2) $A = F - M$, where F is an F_σ and M is meager in X;
(3) $A = (U - B) \cup C$, where U is open and B and C are meager in X;
(4) $A = (F - B) \cup C$, where F is closed and B and C are meager in X;
(5) there is a set M meager in X such that $A - M$ is both open and closed relative to M^c;
(6) $D(A) \cap D(A^c)$ is nowhere dense in X and therefore every nonempty open set contains a point at which either A or A^c is of the first category in X;
(7) $D(A) - A$ is meager in X;
(8) $A \triangle D(A)$ is meager in X;
(9) $A \triangle ID(A)$ is meager in X.

Corollary 7 *(Gleason).*
Let X be a Baire space and let $A, B \in \mathcal{BP}(X)$. Then $ID(A) \cap ID(B) \neq \emptyset \implies A \cap B \neq \emptyset$.

Proof:

Suppose that $A \cap B = \emptyset$ and that the open set $ID(A) \cap ID(B) \neq \emptyset$. But then an elementary computation shows that $\emptyset \neq ID(A) \cap ID(B) \subseteq ((ID(A) - A) \cup A) \cap ((ID(B) - B) \cup B) \subseteq (ID(A) - A) \cup (ID(B) - B)$ is meager by Lemma 6, (9). But $ID(A) \cap ID(B)$ is not meager since X is a Baire space, a contradiction. Hence, $A \cap B \neq \emptyset$. $\square$

Theorem 8 .
Let X be a topological space, let (Y, d) be a metric space and let $\varphi : X \mapsto Y$ be a function that satisfies:
(1) $\varphi^{-1}(B) \in \mathcal{BP}(X)$ for every ball $B \subseteq Y$; and
(2) $\varphi^{-1}(V) \subseteq D(\varphi^{-1}(V))$ if $V \subseteq Y$ is open.
Then the set of points of continuity of φ is comeager in X.

Proof:

Fix $n \geq 1$. We will show that there is an open dense set U_n in X for which φ has oscillation less than or equal to $1/n$ at each point in U_n.

For each $x \in X$, let $V_x \subseteq Y$ be the open ball of d-radius $1/2n$ about $\varphi(x)$. Let $U_x = ID(\varphi^{-1}(V_x))$, and set $U_n = \bigcup_{x \in X} U_x$. U_n is open in X and we claim that U_n is dense in Y. To see this, let $W \subseteq X$ be any nonempty open set and choose $x \in W$. Then $x \in \varphi^{-1}(V_x) \subseteq D(\varphi^{-1}(V_x))$, so $W \cap D(\varphi^{-1}(V_x))$ is nonempty and relatively open in $D(\varphi^{-1}(V_x))$. Since U_x is dense in $D(\varphi^{-1}(V_x))$ by Lemma 3 (14), it follows that $W \cap U_x$ is nonempty. Thus $W \cap U_n$ is nonempty and U_n is dense as required.

Next we show that φ has oscillation less than or equal to $1/n$ at w for every $w \in U_n$. To accomplish this, we will first show that if $V \subseteq Y$ is open and $x \in ID(\varphi^{-1}(V))$, then $\varphi(x) \in \mathrm{cl}_Y(V)$. If not then there is some open neighborhood V' of $\varphi(x)$ which misses V. Since $x \in \varphi^{-1}(V') \subseteq D(\varphi^{-1}(V'))$, we have $ID(\varphi^{-1}(V)) \cap D(\varphi^{-1}(V')) \neq \emptyset$. But $ID(\varphi^{-1}(V'))$ is dense in $D(\varphi^{-1}(V'))$ by Lemma 3 (14), so in fact $ID(\varphi^{-1}(V)) \cap ID(\varphi^{-1}(V')) \neq \emptyset$. Since $\varphi^{-1}(V)$ and $\varphi^{-1}(V')$ have the Baire property, it follows from Corollary 7 that $\varphi^{-1}(V) \cap \varphi^{-1}(V') \neq \emptyset$. This contradicts our assumption that V misses V', so $\varphi(x) \in \mathrm{cl}_Y(V)$.

Now suppose $w \in U_n$. Let $z \in X$ be such that $w \in U_z$. It follows from the above paragraph that for every $x \in U_z$ we have $\varphi(x) \in \mathrm{cl}_Y(V_z)$, where V_z is the ball of radius $1/2n$ about $\varphi(z)$. So $d(\varphi(w), \varphi(x)) \leq d(\varphi(w), \varphi(z)) + d(\varphi(x), \varphi(z)) \leq 1/2n + 1/2n = 1/n$. Since w was arbitrary, φ has oscillation less than or equal to $1/n$ at every point in U_n.

Set $U = \bigcap_{n \geq 1} U_n$. Then U is a countable intersection of dense open sets in X and φ has oscillation 0 at every point in U. So φ is continuous on a comeager set. $\square$

Though Gleason did not formulate Theorem 8, a glance at [5] shows that he had most of the technology in hand to prove it. A word of caution perhaps is in order for readers of [5]: the notation $D(A)$ used here is consistent with that defined in Kuratowski [12], whereas Gleason's notation of $D(A)$ coincides with the $ID(A)$ used here in spite of the fact that he refers to Kuratowski [12] for the properties of his $D(A)$.

The conclusion of Theorem 8 is reminiscent of a property of Baire class 1 functions (a theorem of Baire, [10], Theorem 24.14). However, in general, there is no connection between the functions that satisfy the hypotheses of Theorem 8 and functions of Baire class 1. For example, the function $\varphi = \delta_0$ is a Baire class 1 function, but $\varphi^{-1}(\langle 1/2, 3/2 \rangle) = \{0\}$ is certainly meager in $\mathbb{R}$ and therefore does not satisfy the hypotheses of Theorem 8. On the other hand, let $B = \{(x, y) \mid y > 0\} \cup \mathbb{Q}$ and let $\varphi = \chi_B$, the characteristic function of B. Then φ is Borel measurable and hence is in $\mathcal{BP}(\mathbb{R}^2)$ and if $U \subseteq \mathbb{R}$ is open, then $\varphi^{-1}(U)$ is either empty or contains a nonempty open subset of $\mathbb{R}^2$ and therefore is second category. But φ is not a Baire class 1 function since $\varphi^{-1}(\langle -1/2, 1/2 \rangle) \cap \mathbb{R}$ is the set of irrational numbers, which is not an F_σ.

4   A Strengthening of Gleason's Results

Most, but not all, of the results given in this section are due to Gleason in less general form. They are given here because of their importance in what follows and for the convenience of the reader since Gleason's paper is somewhat obscure. The statements of the results are more general than Gleason's statements and the proofs are somewhat different.

We start with a result on descriptive set theory. It illustrates the power of a transitive group assumption.

Proposition 9 *(Gleason).*
Let X be a Polish space, Y a separable metric space, G a group that acts as a group of homeomorphisms on X and Y and that is transitive on X and let $\varphi : X \mapsto Y$ be $\mathcal{BP}(X)$-measurable and G-equivariant. Then φ is continuous.

Proof:

Let $U \subseteq X$ and $V \subseteq Y$ be open and satisfy $U \cap \varphi^{-1}(V) \neq \emptyset$. Let $x \in U \cap \varphi^{-1}(V)$ so $(x, \varphi(x)) \in U \times V$. If $x' \in X$ choose $g \in G$ such that $g(x) = x'$. Then $(x', \varphi(x')) = (g(x), \varphi(g(x)) = (g(x), g(\varphi(x))) \in g(U) \times g(V)$. Therefore $\mathrm{graph}(\varphi) \subseteq \cup_{g \in G}(g(U) \times g(V))$. Since $\mathrm{graph}(\varphi)) \subseteq X \times Y$ is separable metrizable and therefore Lindelöf there exists a sequence $\{g_n\}_{n \geq 1} \subseteq G$ such that $\mathrm{graph}(\varphi) \subseteq \cup_{n \geq 1}(g_n(U) \times g_n(V))$. Let $X_n = \{x \in X \mid (x, \varphi(x)) \in g_n(U) \times g_n(V)\} = g_n(U \cap \varphi^{-1}(V))$. Since $\cup_{n \geq 1} X_n = X$, some X_n is second category and therefore $U \cap \varphi^{-1}(V)$ is nonmeager in X. Corollary 5 plus Theorem 8 imply that the set of points of continuity of φ in X is residual in X and therefore nonempty. Since G is transitive and φ is G-equivariant, φ is continuous everywhere. $\square$

Gleason did not point out the following corollary (c.f. [4]).

Corollary 10 .
Let X be a Polish space, $x \in X$, and G a Polish group that acts as an abstract transitive group of homeomorphisms of X such that $g \mapsto g(x)$, $G \mapsto X$, is continuous. Let $G_x = \{x \in X \mid g \cdot x = x\}$, a closed subgroup of G. Then the natural G-equivariant mapping $\varphi : gG_x \mapsto g(x)$, $G/G_x \mapsto X$, is a homeomorphism and (G, X) is a topological transformation group that is naturally homeomorphic to the topological transformation group $(G, G/G_x)$.

Proof:

The quotient space G/G_x is a Polish space by a theorem of Hausdorff [7] and the natural G-equivariant mapping $\varphi : gG_x \to g(x)$, $G/G_x \mapsto X$ is a continuous bijection. G acts transitively on both X and G/G_x and $\varphi^{-1} : X \mapsto G/G_x$ is a G-equivariant Borel mapping by Lusin-Souslin's theorem. Proposition 9 now implies that φ^{-1} is continuous and therefore φ is a homeomorphism. $\square$

Corollary 11 *(Gleason).*
Let G be an abstract group and also a Polish space such that, for each fixed $g \in G$, the mapping $h \mapsto gh$, $G \mapsto G$, is continuous and for each fixed $h \in G$, the mapping $g \mapsto gh$, $G \mapsto G$, is $\mathcal{BP}(G)$-measurable. Then G is a Polish group.

Proof:

Fix $h_0 \in G$ and let $\varphi(g) = gh_0$, a $\mathcal{BP}(G)$-measurable mapping. Left translations are continuous by assumption and therefore homeomorphisms since they are invertible. Left translations also act transitively on G. Proposition 9 implies that φ is continuous since $k\varphi(g) = \varphi(kg)$ for all $k, g \in G$. Therefore both left and right translations of G are continuous. The corollary now follows from a well known result of Montgomery [14]. $\square$

Corollary 12 .
Let G be an abstract group and also a Polish space such that, for each fixed $g \in G$, the mapping $h \mapsto hg$, $G \mapsto G$, is continuous and for each fixed $h \in G$, the mapping $g \mapsto hg$, $G \mapsto G$, is $\mathcal{BP}(G)$-measurable. Then G is a Polish group.

Proof:

Let G^* be the group whose underlying set is G with the topology of G but with the multiplication $a * b = ba$. Then G^* is a Polish group and this corollary follows by applying Corollary 12 to G^*. $\square$

Corollary 13 *(Gleason).*
Let G and M satisfy Axiom 1 and Axiom 2(a) for a frame of size one. Then the conclusion to Gleason's Conjecture is true.

Proof:

G acts simply transitively on M since $n = 1$. Fix $m_0 \in M$ and topologize G by requiring that the bijection $g \mapsto g(m_0)$, $G \mapsto M$, be a homeomorphism. G is then an abstract group and a Polish space. $h_n \to h$ if and only if $h_n(m_0) \to h(m_0)$ which implies that $gh_n(m_0) = g(h_n(m_0)) \to g(h(m_0)) = gh(m_0)$ which in turn implies that $gh_n \to gh$ for each $g \in G$, i.e., the mapping $h \mapsto gh$, $G \mapsto G$, is continuous for each $g \in G$.

On the other hand, fix $h \in G$. The graph of the mapping $g \mapsto gh$, $G \mapsto G$, is homeomorphic to $\{(g(m_0), g(h(m_0))) \mid g \in G\}$, an analytic set. From this it easily follows that the mapping $g \mapsto gh$, $G \mapsto G$, is $\mathcal{BP}(G)$-measurable. The present corollary now follows from Corollary 12. $\square$

5 The General Case

We first start with some basic properties of analytic frames.

Lemma 14 .
Let M be a Polish space, G an abstract group of homeomorphisms of M, I and J nonempty finite or countably infinite disjoint index sets, $(p_i)_{i\in I}$ a frame for G acting on M and $(q_j)_{j\in J}$ a tuple of elements of M. Let $A = \{(g(p_i))_{i\in I} \mid g \in G\}$, $C(r) = \{(g(p_i))_{i\in I} \oplus (g(r)) \mid g \in G\}$ $(r \in M)$ and $B = \{(g(p_i))_{i\in I} \oplus (g(q_j))_{j\in J} \mid g \in G\}$ and suppose that $C(r)$ is an analytic set for every $r \in M$. Then A is analytic set and B is an analytic set Borel isomorphic to A. If A is a Borel set, then B is a Borel set. If A is a G_δ-set, then B is a G_δ-set homeomorphic to A.

Proof:

$\quad$ A is an analytic set since it is the continuous image of any $C(r)$. If $j_0 \in J$ then $C_{j_0} = \{(g(p_i))_{i\in I} \oplus (g(q_{j_0})) \mid g \in G\} \oplus \prod_{j\in J-\{j_0\}} M$ is an analytic subset of $M^{I\cup J}$ and therefore $B = \cap_{j\in J} C_{q_j}$ is an analytic set since the intersection of a sequence of analytic sets is analytic. The natural projection of B onto A is a continuous bijection since $(p_i)_{i\in I}$ is a frame for G acting on M and therefore is a Borel isomorphism by [13], Theorem 4.2. We are now done in the analytic and Borel cases.

$\quad$ Finally, suppose A is a G_δ. Let $\varphi : A \mapsto B$ be given by $\varphi : (g(p_i))_{i\in I} \mapsto (g(p_i))_{i\in I} \oplus (g(q_j))_{j\in J}$, the inverse of projection of B onto A and therefore a Borel mapping. Then A is a Polish space, B is a separable metric space and φ is a G-equivariant mapping. Proposition 9 implies that φ is continuous. Since we have already noted that the natural projection of B onto A, viz. φ^{-1}, is continuous, we have that φ is a homeomorphism. Hence, B is a G_δ since it is a Polish space. $\square$

Corollary 15 .
If Axiom 2(c) holds, then Axiom 2(d) holds and Axiom 2(d) holds if and only if Axiom 2(e) holds.

Proof:

$\quad$ Call a frame F a G_δ-frame if its orbit under G is a G_δ-set. Any augmentation of a finite or countably infinite G_δ-frame for the action of G on M by a finite or countably infinite number of elements of M is again a G_δ-frame for the action of G on M by Lemma 14. $\square$

Theorem 16 .
Let G and M satisfy Axiom 1 and Axiom 2(c) or Axiom 2(d) or Axiom 2(e). Then the conclusion of Gleason's Conjecture is true.

Proof:

$\quad$ Corollary 15 implies that it suffices to give the proof under the assumption that Axiom 2(e) holds.

$\quad$ Suppose that $\{m_\ell\}_{\ell\geq 1}$ is a dense sequence in M such that $\mathcal{O} = \{(g(m_1), g(m_2), \dots) \mid g \in G\} \subseteq M^{\mathbb{N}}$ is a G_δ subset of $M^{\mathbb{N}}$ and that $\{(g(m_1), g(m_2), \dots, g(q)) \mid g \in G\} \subseteq M^{\mathbb{N}} \times M$ is a Borel subset of $M^{\mathbb{N}} \times M$ set for each $q \in M$. The natural diagonal action of G on $M^{\mathbb{N}}$ is an abstract group of homeomorphisms of $M^{\mathbb{N}}$. If $F = (m_1, m_2, \dots) \in M^{\mathbb{N}}$ is the frame and $Q = (q_1, q_2, \dots) \in \mathcal{O}$, then, with obvious notation, the set $\{(g(F), g(Q)) \mid g \in G\} \subseteq \mathcal{O} \times \mathcal{O}$ is a Borel set by Lemma 14. $\mathcal{O}$ is a Polish space since it is a G_δ-subset of a Polish space and G acts as a simply transitive group of homeomorphisms of $\mathcal{O}$. Then the pair G and $\mathcal{O}$ satisfy Axiom 1 and Axiom 2(a). Therefore the pair $(G, \mathcal{O})$ can be made into a Polish topological transformation group by Corollary 13.

$\quad$ If $x \in M$ choose $h \in G$ such that $x = h(m_1)$. Now the mapping $g \mapsto gh \mapsto gh(F) \mapsto gh(m_1) = g(x)$ is continuous for every $x \in M$ and G-equivariant. Corollary 10 now implies that the pair (G, M) is a transitive

Polish topological transformation group. □

The next simple general result implies that the topology on G determined by Theorem 16 is unique.

Proposition 17 .
Let M be a Polish space and let G be an abstract group of homeomorphisms of M. If $\mathfrak{T}_1$ and $\mathfrak{T}_2$ are two Polish group topologies on G such that both $((G, \mathfrak{T}_1), M)$ and $((G, \mathfrak{T}_2), M)$ are topological transformation groups, then $\mathfrak{T}_1 = \mathfrak{T}_2$.

Proof:

Let F be a frame whose coordinates are dense in M. The mapping $G \mapsto G \cdot F \subset M^{\mathbb{N}}$, $g \mapsto g \cdot F$, is a bijection. Hence, the mappings $(G, \mathfrak{T}_\ell) \mapsto G \cdot F$, $g \mapsto g \cdot F$ ($\ell = 1, 2$) are Borel isomorphisms since they are continuous. Therefore, the group isomorphism $(G, \mathfrak{T}_1) \mapsto (G, \mathfrak{T}_2)$, $g \mapsto g$, is a Borel mapping and therefore a topological isomorphism. □

The counterexample given in Section 2 strongly suggests that the sufficient G_δ G-orbit of a frame condition in the axioms cited in Theorem 16 cannot be omitted. Unfortunately this condition is not necessary, as the following proposition demonstrates.

Proposition 18 .
There exist a Polish group G and a Polish space X such that (G, X) is a transitive topological transformation group with a frame such that the following property holds: the orbit $G \cdot x$ is not a G_δ set in X^n for every frame $x \in X^n$ ($n \in \mathbb{N} \cup \{\infty\}$).

Proof:

Let X be the reals, let A be the multiplicative group of positive rationals with the discrete topology, let B be the additive group of the reals and let $G = B \rtimes A$, the natural semidirect product of B and A. (G, X) is a transitive topological transformation group and $(1, -1)$ is a frame for G in X^2. Let $n \in \mathbb{N}$ and let $x \in X^n$ be a frame for G. Then $x \neq 0 \in X^n$. Suppose that $G \cdot x$ is a G_δ subset of X^n. Then $G \cdot x$ is a Polish space and the mapping $g \mapsto gx$ is a homeomorphism by Corollary 10. It follows that $A \cdot x$ is a G_δ in X^n since A closed in G implies $A \cdot x$ is closed in $G \cdot x$. But $A \cdot x = \{qx \mid q \in \mathbb{Q}, q > 0\}$ is therefore a G_δ in $\{qx \mid q \in \mathbb{R}, q > 0\}$, a contradiction since the positive rationals are not a G_δ-subset of the positive reals. □.

Recall the following theorem of Becker-Kechris.

Theorem 19 ([2] Theorem 5.1.5) .
Let G be a Polish group, let X be a Polish G-space, and let $E \subseteq X$ be a G-invariant Borel set. There exists a Polish topology finer than the original topology of X (and thus having the same Borel structure) in which E is now open and the action of G on X is still continuous.

The following proposition shows in a rather strong manner that the assumption that G is a Polish group cannot be omitted from the Becker-Kechris Theorem 19.

Theorem 20 .
There exists a separable metrizable topological group G, a Polish G-space X and a G-invariant K_σ-subset $E \subseteq X$ such that there is no finer Polish topology on X which makes E a G_δ and such that the action of G on E is still continuous.

Proof:

Let G be as in Proposition 2, let $M = \mathbb{R}$ and let $X = M^2$. G is a separable metrizable K_σ group and the pair (G, M) is a topological transformation group. The orbit of any frame for G in M^n and therefore in X^n is a K_σ. $(1, -1) \in X$ is a frame for G, $E = G \cdot (1, -1) \subseteq X$ is a K_σ, G acts simply transitively on E and $G \cdot (x, q) \subseteq X^2$ is K_σ for every $x, q \in X$. Suppose that there is a finer Polish topology on X that makes E into a G_δ and such that the action of G on X is still continuous. This new topology and the original topology on X generate the same Borel sets and therefore $G \cdot (x, q)$ is still a Borel set for every $x, q \in X$. Then the hypotheses of Corollary 13 are satisfied and the abstract group G can be made into a Polish group such that the pair (G, X) is a Polish topological transformation group. But G cannot be given any Polish group topology by Proposition 2, a contradiction. $\square$

6 A Corollary on Lie Groups and Manifolds

The following corollary is obviously motivated by Gleason [5], Corollary 2, who gave a terse indication of its proof. Perhaps this is the result desired by Gleason.

Corollary 21 .
Let G and M satisfy Axiom 1 and Axiom 2(c). In addition assume that M is of finite dimension and the G-orbit $G \cdot F$ is locally connected at one point. Then (G, M) can be made into a Polish topological transformation group such that G is a Lie group and M is a manifold homeomorphic with a quotient group of G.

Proof:

(G, M) can be made into a Polish topological transformation group by Theorem 16. M^n is of finite dimension ([9], Theorem III 4, The Product Theorem, p. 33) and therefore the G-orbit $G \cdot F \subseteq M^n$ is of finite dimension ([9], Theorem III 1, p. 26). Since G-orbit is locally connected at one point, it is locally connected since G acts transitively on the orbit. Therefore G is a finite dimensional locally connected Polish group since it is homeomorphic to the G-orbit $G \cdot F$. If U is a connected open subset of G, then U is a connected, locally connected complete metrizable space and therefore U is arcwise connected ([8], Theorem 3-17). Hence, G is a finite dimensional locally arcwise connected Polish group and therefore is a Polish Lie group ([6], Theorem 7.2). Finally, if $x \in M$ and G_x is the G-stability group at x, then G_x is a closed subgroup of G, G/G_x is a manifold ([18], Theorem 3.58) and M is homeomorphic to G/G_x by Corollary 10. □

As a final comment, Gleason proved the following corollary.

Corollary 22 *(Gleason [5], Corollary 3).*
Let G be a topological group acting continuously and effectively on a complete separable metric space M. Let T be an analytic subgroup of G (that is, a subgroup which is the continuous image of the set of irrational numbers) which is simply transitive on M. Then T is closed.

As Gleason notes in his proof, we have $G = G_x T$, where G_x is the stability group at $x \in M$, a closed subgroup of G, and $G_x \cap T = \{e\}$. Therefore Proposition 5 and Corollary 6 of [1] provide more general results, at least in the case for which (G, M) is a Polish transformation group. The proofs of this proposition and corollary appear to have nothing in common with the techniques employed by Gleason.

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