THE DESCRIPTIVE COMPLEXITY OF SERIES REARRANGEMENTS

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Abstract. We consider the descriptive complexity of some subsets of the infinite permutation group S_∞ which arise naturally from the classical series rearrangement theorems of Riemann, Levy, and Steinitz. In particular, given some fixed conditionally convergent series of vectors in Euclidean space $\mathbb{R}^d$, we study the set of permutations which make the series diverge, as well as the set of permutations which make the series diverge properly. We show that both collections are $\mathbf{\Sigma}^0_3$-complete in S_∞, regardless of the particular choice of series.

1. Introduction

The goal of this paper is to establish the exact descriptive complexity of some interesting subsets of the Polish group S_∞ of permutations of ω, endowed with the topology of pointwise convergence on ω, considered as a discrete set. Our methods will involve a blending of the techniques of classical real analysis and geometry, with the descriptive set theoretic notion of continuous reducibility between Polish spaces.

First we recall Bernhard Riemann's celebrated rearrangement theorem of 1876 [6], now a staple of every graduate course in real analysis, which states the following remarkable fact (presented here as in [8]): given a conditionally convergent series of real numbers $\sum_{k=0}^{\infty} a_k$, and two extended real numbers $\alpha, \beta \in [-\infty, \infty]$ with $\alpha \leq \beta$, it is possible to find an infinite permutation $\pi \in S_\infty$ for which $\liminf_{n\to\infty} \sum_{k=0}^{n} a_{\pi(k)} = \alpha$ and $\limsup_{n\to\infty} \sum_{k=0}^{n} a_{\pi(k)} = \beta$. In other words, by varying one's choice of α and β, it is possible to rearrange the terms of a conditionally convergent infinite series so that the partial sums converge to any particular real number, or diverge to plus or minus infinity, or even diverge properly.

Almost as famous as Riemann's original theorem is the following d-dimensional analogue:

Levy-Steinitz Theorem. *Let $\sum_{k=0}^{\infty} v_k$ be a conditionally convergent series of vectors in $\mathbb{R}^d$. Then there exists an affine subspace $A \subseteq \mathbb{R}^d$ (that is, a space of the form $A = v + M$ where $v \in \mathbb{R}^d$ and $M \subseteq \mathbb{R}^d$ is a linear subspace) such that whenever $a \in A$, there is $\pi \in S_\infty$ with $\sum_{k=0}^{\infty} v_{\pi(k)} = a$.*

The statement above implies that the set of all possible sums of rearrangements of a conditionally convergent series of d-dimensional vectors is at least as rich as in the 1-dimensional case. An incomplete proof was first given by Levy in 1905 [5], and the complete proof was furnished by Steinitz in 1913 [9]. Steinitz's proof, which is the one more commonly seen today, relied on a particular geometric constant for Euclidean spaces which is now commonly called the *Steinitz constant*. The proof is nontrivial, and an excellent concise version of it may be found in the paper [7] by P. Rosenthal. Our proofs will also rely heavily on the existence of a Steinitz constant.

The Levy-Steinitz theorem gives rise to a natural partition of S_∞, into the set $\mathcal{D}$ of all permutations π for which $\sum_{k=0}^{\infty} v_{\pi(k)}$ diverges (either properly or to ∞, where ∞ denotes the point at infinity in the one-point compactification of $\mathbb{R}^d$), and the complement set $S_\infty \backslash \mathcal{D}$ of permutations π for which $\sum_{k=0}^{\infty} v_{\pi(k)}$ converges to some vector in $\mathbb{R}^d$. Both $\mathcal{D}$ and its complement are interesting nontrivial sets. For instance, it is easy to observe, as we will do briefly in Section 4, that both $\mathcal{D}$ and $S_\infty \backslash \mathcal{D}$ are uncountable and dense in S_∞, and also that $\mathcal{D}$ is a comeager set.

We wish to examine these collections from the vantage point of descriptive set theory, or, loosely speaking, the study of the definable subsets of Polish spaces. *Definable* here may refer to Borel, analytic, projective, or any other class of "well-behaved" sets, which are typically closed under continuous preimages. Of course the Borel sets may be stratified by their relative complexity into a *Borel hierarchy* indexed by the countable ordinals, whose exact definition we will recall for the reader in Section 2. It is an empirical phenomenon that a great bulk of those Borel sets which present themselves in the everyday study of mathematics will fall into the very bottom few levels of the Borel hierarchy. Thus there has been some industry for descriptive set theorists in finding "natural" examples of Borel sets which are "more complex" than usual. For some instances of such sets, the reader may consult the well-known references [2] and Sections 23, 27, 33, and 37 of [4], or the paper [1], which produces many examples in the field of ordinary differential equations.

Our objective here will be to establish the exact descriptive complexity of our set $\mathcal{D}$ and its complement. In classical terminology, we will show that $\mathcal{D}$ is $G_{\delta\sigma}$ but not $F_{\sigma\delta}$ (and hence not F_σ, G_δ, open, nor closed). Using the more modern notation, we prove:

Theorem 1. *Let* $\sum_{k=0}^{\infty} v_k$ *be any conditionally convergent series of vectors in* $\mathbb{R}^d$, *and let* $\mathcal{D} \subseteq S_\infty$ *be the set of all permutations* π *for which* $\sum_{k=0}^{\infty} v_{\pi(k)}$ *diverges. Then* $\mathcal{D}$ *is* $\mathbf{\Sigma}^0_3$*-complete.*

Of course, it follows immediately that $S_\infty \backslash \mathcal{D}$ is $\mathbf{\Pi}^0_3$-complete. Now, for $\pi \in S_\infty$, say that the rearrangement $\sum_{k=0}^{\infty} v_{\pi(k)}$ *diverges properly* if the series diverges, but does not diverge to infinity. Our methods also give the following result:

Theorem 2. *Let* $\sum_{k=0}^{\infty} v_k$ *be any conditionally convergent series of vectors in* $\mathbb{R}^d$*, and let* $\mathcal{DP} \subseteq S_\infty$ *be the set of all permutations* π *for which* $\sum_{k=0}^{\infty} v_{\pi(k)}$ *diverges properly. Then* $\mathcal{DP}$ *is* $\mathbf{\Sigma}^0_3$*-complete.*

It follows that the set $S_\infty \backslash \mathcal{DP}$ of series rearrangements which either converge to a vector in $\mathbb{R}^d$, or which diverge to ∞, is also a $\mathbf{\Pi}^0_3$-complete set in S_∞. Notice that, remarkably, none of the above statements depend on the nature of the particular conditionally convergent series $\sum_{k=0}^{\infty} v_k$ that we choose! Thus, we exhibit continuum-many sets in S_∞ which lie no lower on the Borel hierarchy than the third level.

2. Definitions and terminology

First we recall the definition of the Borel hierarchy. Given a Polish space X, we let $\mathbf{\Sigma}^0_1(X)$ be the family of all open subsets of X, and $\mathbf{\Pi}^0_1(X)$ the family of all closed subsets of X. We set $\mathbf{\Delta}^0_1(X) = \mathbf{\Sigma}^0_1(X) \cap \mathbf{\Pi}^0_1(X)$, so $\Delta^0_1(X)$ consists of the clopen sets in X. The rest of the family is defined recursively as follows: Suppose for some countable ordinal β, we have defined the classes $\mathbf{\Sigma}^0_\alpha(X)$ and $\mathbf{\Pi}^0_\alpha(X)$ for all $\alpha < \beta$. Then we set

$$\begin{gathered}\mathbf{\Sigma}^0_\beta(X) = \{\textstyle\bigcup_{n\in\omega} A_n : A_n \in \mathbf{\Pi}^0_{\alpha_n} \text{ for some } \alpha_n < \beta\},\\ \mathbf{\Pi}^0_\beta(X) = \{A^c : A \in \mathbf{\Sigma}^0_\beta(X)\}, \text{ and}\\ \mathbf{\Delta}^0_\beta(X) = \mathbf{\Sigma}^0_\beta(X) \cap \mathbf{\Pi}^0_\alpha(X).\end{gathered}$$

It is well known that $\mathbf{\Delta}^0_\beta(X) \subseteq \mathbf{\Sigma}^0_\beta(X), \mathbf{\Pi}^0_\beta(X) \subseteq \mathbf{\Delta}^0_{\beta+1}$ for each β, and that the inclusions are all proper.

Let X, Y be Polish spaces and $A \subseteq X$, $B \subseteq Y$. If there exists a continuous function $f : X \to Y$ such that $f^{-1}(B) = A$, then we say that A is *Wadge reducible* or *continuously reducible* to B, and we write $A \leq_W B$. Intuitively, we think that A is "no more complex" than B.

Let $\mathbf{\Gamma}$ be any of the pointclasses $\mathbf{\Sigma}^0_\beta$, $\mathbf{\Pi}^0_\beta$, or $\mathbf{\Delta}^0_\beta$. A standard inductive argument through the hierarchy shows that $\mathbf{\Gamma}$ is closed under continuous preimages, i.e., whenever X and Y are Polish, $A \subseteq X$, $B \in \mathbf{\Gamma}(Y)$, and A is continuously reducible to B, then we have $A \in \mathbf{\Gamma}(X)$.

The above comment provides a useful tool for determining the complexity of a set. We say that a subset B of a Polish space Y is $\mathbf{\Gamma}$-*hard* if for every Polish space X and every $A \in \mathbf{\Gamma}(X)$ we have $A \leq_W B$. It follows from the above comments that if B is $\mathbf{\Gamma}$-hard, then $\mathbf{\Gamma}$ is a lower bound for the descriptive complexity of B. If in addition we have $B \in \mathbf{\Gamma}(Y)$, then we say that B is $\mathbf{\Gamma}$-*complete*, and we have determined its exact complexity in the Borel hierarchy.

The most common method for showing that a set B is $\mathbf{\Gamma}$-hard is to find a set A which is already known to be $\mathbf{\Gamma}$-complete, and prove that $A \leq_W B$ by constructing an explicit continuous reduction. This will be the method of our proof in Section 4, and we will make use of the following subset $\mathcal{C}$ of the Baire space ω^ω:

$$\mathcal{C} = \{x \in \omega^\omega : \lim_{n\to\infty} x(n) = \infty\}.$$

Exercise 23.2 of [4] asks the reader to show that $\mathcal{C}$ is in fact $\mathbf{\Pi}^0_3$-complete. It necessarily follows that the complement $\omega^\omega \backslash \mathcal{C}$ is $\mathbf{\Sigma}^0_3$-complete. Our proof will continuously reduce this complement $\omega^\omega \backslash \mathcal{C}$, simultaneously, to both $\mathcal{D}$ and $\mathcal{DP}$, and thus establish the $\mathbf{\Sigma}^0_3$-hardness of the latter two sets.

We regard each nonnegative integer n as a von Neumann ordinal, i.e. we think of each n as the set $\{0, ..., n-1\}$. If a function $\pi : n \to \omega$ is injective, then we call π a *finite partial permutation.* We use the notation $\mathrm{dom}(\pi)$ to refer to the map's domain $n = \{0, ..., n-1\}$ and $\mathrm{ran}(\pi)$ to refer to its range $\{\pi(0), ..., \pi(n-1)\}$. If $\sigma \in S_\infty$ or if σ is a finite partial permutation, then we say σ *extends* π if $\sigma \restriction \mathrm{dom}(\pi) = \pi$.

3. The Bounded Walk lemma

In this section we will develop the main technical lemma on which our proof is built. An intuitive explanation for the Bounded Walk lemma is as follows: Consider a conditionally convergent series as an abstract infinite collection $(v_k)_{k\in\omega}$ of vectors in $\mathbb{R}^d$ from which we may build finite paths. Let α and β be two points in $\mathbb{R}^d$. Suppose we have already walked very close to α and we now wish to walk very close to β. Then if all the remaining vectors to choose from are sufficiently small (say less than $\rho \cdot \frac{1}{C_d}$ where ρ and C_d are some constants to be determined later), and α and β are sufficiently far apart (say further than 3ρ), then it is possible to build a finite path which (1) extends the path we have already walked; (2) uses up all except arbitrarily small remaining vectors; (3) takes us arbitrarily close to β; and (4) does not wander arbitrarily far from the straight-line path connecting α and β. In addition we may (5) use up any particular vector we wish. (Note that conditions (1) and (2) allow us to repeat this "bounded walk" process between as many points as we like, as often as we like.)

Now we aim to establish such a lemma. Before we do so, we first recall the following classical result as stated in [7], which is attributed to Steinitz, and which asserts the existence of a very useful "bounded rearrangement constant" C_d in Euclidean space, now referred to as the Steinitz constant:

Lemma 3 (Polygonal Confinement Theorem)**.** *Let $d \geq 1$ be any integer. Then there exists a constant C_d which satisfies the following statement: Whenever $v_0, v_1, ..., v_m$ are vectors in $\mathbb{R}^d$ which sum to 0 and satisfy $||v_i|| \leq 1$ for each $i \leq m$, then there is a finite permutation $P \in S_m$ with the property that*

$$\left|\left| v_0 + \sum_{i=1}^{j} v_{P(i)} \right|\right| \leq C_d$$

for every j.

The Polygonal Confinement Theorem is the basis for the remaining lemmas in this section.

Lemma 4. *Let $\alpha, v_1, ..., v_m$ be vectors in $\mathbb{R}^d$ which sum to $\beta \in \mathbb{R}^d$, let $\rho > 0$, and let C_d be as in the Polygonal Confinement Theorem. Further suppose we have $||v_i|| \leq \rho \cdot \frac{1}{C_d}$ for each $i \leq m$ and $||\beta - \alpha|| \geq \rho$. Then there is a finite permutation $P \in S_m$ with the property that*

$$\left|\left|\sum_{i=1}^{j} v_{P(i)}\right|\right| \leq 2||\beta - \alpha||$$

for every j.

Proof. Without loss of generality we may assume $\alpha = 0$, for if not, replace α with 0 and β with $\beta - \alpha$. We may also without loss of generality take $\rho = C_d$, for if not, replace v_i with $v_i \cdot \dfrac{C_d}{\rho}$ and β with $\beta \cdot \dfrac{C_d}{\rho}$. In this case we have $||v_i|| \leq 1$ for each i and $||\beta|| \geq C_d$.

Now let s be an integer sufficiently large so that $\frac{||\beta||}{s} \leq 1$, and set $v_{m+1} = v_{m+2} = ... = v_{m+s} = -\beta/s$. Then $\alpha, v_1, ..., v_m, v_{m+1}, ..., v_{m+s}$ are a collection of vectors which satisfy the hypotheses of the Polygonal Confinement Theorem, and hence there exists a permutation $P' \in S_{m+s}$ for which

$$\left|\left|\alpha + \sum_{i=1}^{j'} v_{P'(i)}\right|\right| = \left|\left|\sum_{i=1}^{j'} v_{P'(i)}\right|\right| \leq C_d$$

for every $j' \leq m+s$. Let $P \in S_m$ be the unique permutation which arranges $1, ..., m$ in the same order as P'.

Now let $j \leq m$ be arbitrary. Let $j' \geq j$ be the least integer for which $\{P(1), ..., P(j)\} \subseteq \{P'(1), ..., P'(j')\}$. Note that since P and P' arrange $1, ..., j$ in the same order, then for any $i \leq j'$, we must have either $(P')^{-1}(i) \in \{1, ..., j\}$ or $(P')^{-1}(i) \in \{m+1, ..., m+s\}$. Let $I = \{i \leq j' : P^{-1}(i) \in \{m+1, ..., m+s\}\}$. Then we have:

$$\begin{aligned}
\left|\left|\sum_{i=1}^{j} v_{P(i)}\right|\right| &= \left|\left|\sum_{i=1}^{j'} v_{P'(i)} - \sum_{i\in I} v_i\right|\right| \\
&\leq \left|\left|\sum_{i=1}^{j'} v_{P'(j)}\right|\right| + \sum_{i\in I} ||v_i|| \\
&\leq C_d + \sum_{i\in I} \frac{||\beta||}{s} \\
&\leq ||\beta|| + s \cdot \frac{||\beta||}{s} \\
&= 2||\beta||
\end{aligned}$$

as required. □

Lemma 5. *Let σ be any finite partial permutation of ω, and let $\sum_{k=0}^{\infty} v_k$ be a series of vectors. If $\pi \in S_\infty$ is any permutation for which $\sum_{k=0}^{\infty} v_{\pi(k)}$ converges, then there exists another permutation $\pi' \in S_\infty$ for which*

(1) π' *extends* σ*, and*

(2) $\sum_{k=0}^{\infty} v_{\pi'(k)} = \sum_{k=0}^{\infty} v_{\pi(k)}$.

Proof. This may be accomplished by simply finding a finitely supported permutation $\tau \in S_\infty$ for which $\tau \circ \pi \restriction \mathrm{dom}(\sigma) = \sigma$, and setting $\pi' = \tau \circ \pi$. □

Lemma 6 (Bounded Walk Lemma)**.** *Let* $\sum_{k=0}^{\infty} v_k$ *be a conditionally convergent series of vectors in* $\mathbb{R}^d$*. Let* $\rho > 0$ *and* $\epsilon > 0$ *be arbitrary and let* C_d *be as in the Polygonal Confinement Theorem. Let* $n \in \omega$ *be arbitrary. Suppose* π *is a finite partial permutation with* $\mathrm{dom}(\pi) = J+1 \in \omega$ *and such that* $||v_k|| < \rho \cdot \frac{1}{C_d}$ *whenever* $k \notin \mathrm{ran}(\pi)$*. Further suppose that* $\alpha \in \mathbb{R}^d$ *satisfies* $\left|\left|\alpha - \sum_{k=0}^{J} v_{\pi(k)}\right|\right| < \rho$*, and* $\beta \in \mathbb{R}^d$ *satisfies* $||\beta - \alpha|| \geq 3\rho$*, and* $\sum_{k=0}^{\infty} v_{\tau(k)} = \beta$ *for some* $\tau \in S_\infty$*.*

Then there exists a finite partial permutation σ *with* $\mathrm{dom}(\sigma) = I+1 \in \omega$ *which satisfies the following properties:*

(1) σ *extends* π*;*

(2) $||v_k|| < \epsilon \cdot \frac{1}{C_d}$ *whenever* $k \notin \mathrm{ran}(\sigma)$*;*

(3) $\left|\left|\beta - \sum_{k=0}^{I} v_{\sigma(k)}\right|\right| < \epsilon$*;*

(4) $\left|\left|\sum_{k=J+1}^{i} v_{\sigma(k)}\right|\right| \leq 6||\beta - \alpha||$ *whenever* $J+1 \leq i \leq I$*; and*

(5) $n \in \mathrm{ran}(\sigma)$*.*

Proof. By applying Lemma 5, we may assume without loss of generality that the permutation τ for which $\sum_{k=0}^{\infty} v_{\tau(k)} = \beta$ is also such that τ extends π. Choose $I \in \omega$ to be so large that $\tau^{-1}(n) \leq I$, $\left|\left|\beta - \sum_{k=0}^{I} v_{\tau(k)}\right|\right| < \min(\rho, \epsilon)$, and that $||v_{\tau(k)}|| < \epsilon \cdot \frac{1}{C_d}$ for all $k > I$.

Set $\alpha_1 = \sum_{k=0}^{J} v_{\tau(k)} = \sum_{k=0}^{J} v_{\pi(k)}$ and set $\beta_1 = \sum_{k=0}^{I} v_{\tau(k)}$. Now notice that we have

$$\begin{aligned}||\beta_1-\alpha_1|| &= ||\beta-\alpha-(\beta-\beta_1)+(\alpha-\alpha_1)|| \\ &\geq ||||\beta-\alpha||-||\beta-\beta_1||-||\alpha-\alpha_1|||| \\ &\geq ||3\rho-\rho-\rho|| \\ &= \rho.\end{aligned}$$

Moreover the images $\tau(J+1),...,\tau(I)$ do not lie in the range of π, since τ is a bijection extending π and $\mathrm{dom}(\pi)=\{0,...,J\}$. Hence $v_{\tau(J+1)},...,v_{\tau(I)}$ all have length less than $\rho\cdot\frac{1}{C_d}$ by our hypothesis, and therefore we may apply Lemma 4 to find a bijection $P:\{\tau(J+1),...,\tau(I)\}\to\{\tau(J+1),...,\tau(I)\}$ which satisfies

$$\left|\left|\sum_{k=J+1}^{i} v_{P(\tau(k)}\right|\right| \leq 2||\beta_1-\alpha_1||$$

whenever $J+1\leq i\leq I$. If we define $\sigma: I+1\to\omega$ by $\sigma(k)=\tau(k)$ for $k\leq J$ and $\sigma(k)=P(\tau(k))$ for $J<k\leq I$, then σ is a finite partial permutation with domain $I+1$ which clearly satisfies (1) above.

Note that if $k\notin\mathrm{ran}(\sigma)$, then $k\notin\{\sigma(0),...,\sigma(J),\sigma(J+1),...,\sigma(I)\}=\{\tau(0),...,\tau(J),P(\tau(J+1)),...,P(\tau(I)\}=\{\tau(0),...,\tau(I)\}$. So $\tau^{-1}(k)>I$, and hence by our choice of I, we have $||v_k||=||v_{\tau(\tau^{-1}(k))}||<\epsilon\cdot\frac{1}{C_d}$. Thus (2) is also satisfied.

Since P is a bijection, we have

$$\begin{aligned}\left|\left|\beta-\sum_{k=0}^{I} v_{\sigma(k)}\right|\right| &= \left|\left|\beta-\left(\sum_{k=0}^{J} v_{\tau(k)}+\sum_{k=J+1}^{I} v_{P(\tau(k)}\right)\right|\right| \\ &= \left|\left|\beta-\sum_{k=0}^{I} v_{\tau(k)}\right|\right| < \epsilon,\end{aligned}$$

so (3) is satisfied.

Lastly note that $||\beta_1-\alpha_1||\leq||\beta-\beta_1||+||\beta-\alpha||+||\alpha-\alpha_1||<\rho+||\beta-\alpha||+\rho\leq 3||\beta-\alpha||$ by our hypothesis. Hence, we have

$$\begin{aligned}\left|\left|\sum_{k=J+1}^{i} v_{\sigma(k)}\right|\right| &= \left|\left|\sum_{k=J+1}^{i} v_{P(\tau(k))}\right|\right| \\ &\leq 2||\beta_1-\alpha_1|| \\ &\leq 6||\beta-\alpha||\end{aligned}$$

whenever $J+1\leq i\leq I$. So (4) is also satisfied. (5) holds simply because $\tau^{-1}(n)\leq I=\mathrm{dom}(\tau)$, and hence $n\in\mathrm{ran}(\tau)=\mathrm{ran}(\sigma)$. So the lemma is proved. ☐

Remark. For simplicity, from now on when we apply the Bounded Walk lemma, we will say that we use it to *walk from α to β*, where α and β are as in the statement of the theorem.

4. Proof of Theorems 1 and 2

First let us make a few simple observations. We will describe our sets of interest using logical notation, with the assumption in place that all quantified variables range over ω. First notice that by Cauchy's criterion for convergence, the following equivalence holds:

$$\pi \in \mathcal{D} \;\leftrightarrow\; \exists m\ \forall n\ \exists i\ \exists j\ \left[i, j \geq n \wedge \left\|\sum_{k=i}^{j} v_{\pi(k)}\right\| \geq \tfrac{1}{m}\right].$$

Since the latter predicate is an open condition in S_∞, a count of quantifiers verifies that $\mathcal{D}$ indeed lies in $\mathbf{\Sigma}^0_3(S_\infty)$. Now fix any $m \geq 1$, and consider the set $\mathcal{D}_m$ defined by the following rule:

$$\pi \in \mathcal{D}_m \;\leftrightarrow\; \forall n\ \exists i\ \exists j\ \left[i, j \geq n \wedge \left\|\sum_{k=i}^{j} v_{\pi(k)}\right\| \geq \tfrac{1}{m}\right].$$

Then $\mathcal{D}_m$ is a nonempty $\mathbf{\Pi}^0_2$ (G_δ)-subset of $\mathcal{D}$ which is invariant under multiplication by finitely supported permutations, and hence dense in S_∞. This shows that $\mathcal{D}$ is a comeager set. The complement $S_\infty \backslash \mathcal{D}$ is also nonempty and invariant under finitely supported permutations, and hence dense as well.

In fact, if we let $T \subseteq S_\infty$ be the set of all permutations whose only action is to transpose (perhaps infinitely many) consecutive integers, then it is easy to see that T is uncountable, and that both $\mathcal{D}$ and $S \backslash \mathcal{D}$ are invariant under multiplication by elements of T. Thus both sets are uncountable dense, i.e. in some sense they are "large" nontrivial sets in S_∞, as promised in the introduction. The reader may consult [3] for similar observations about some other sets in S_∞ which are closely related to our $\mathcal{D}$ and $\mathcal{DP}$.

Next define a set $\mathcal{I} \subseteq S_\infty$ by the rule

$$\pi \in \mathcal{I} \;\leftrightarrow\; \exists m\ \forall n\ \exists i\ \left[i \geq n \wedge \left\|\sum_{k=0}^{i} v_{\pi(k)}\right\| \leq m\right].$$

Then $\mathcal{I}$ is a $\mathbf{\Sigma}^0_3$ set, which consists exactly of those permutations whose corresponding series rearrangements $\displaystyle\sum_{k=0}^{\infty} v_{\pi(k)}$ do not diverge to infinity. Since $\mathcal{DP} = \mathcal{D} \cap \mathcal{I}$, so too we have $\mathcal{DP} \in \mathbf{\Sigma}^0_3(S_\infty)$.

Proof of Theorems 1 and 2. In light of our comments above, it suffices to show that $\mathcal{D}$ and $\mathcal{DP}$ are $\mathbf{\Sigma}^0_3$-hard. Recall that the set $\mathcal{C} = \{x \in \omega^\omega : \lim_{n\to\infty} x(n) = \infty\}$ is known to be $\mathbf{\Sigma}^0_3$-complete. We will build a function $f : \omega^\omega \to S_\infty$ that will be a continuous reduction from $\omega^\omega \backslash \mathcal{C}$ to both $\mathcal{D}$ and $\mathcal{DP}$ simultaneously. That is, both of the following will hold:

$$x \in \omega^\omega \backslash \mathcal{C} \leftrightarrow f(x) \in \mathcal{D},$$

$$x \in \omega^\omega \backslash \mathcal{C} \leftrightarrow f(x) \in \mathcal{DP}.$$

Fix an arbitrary $x \in \omega^\omega$. Let $\boldsymbol{v} = \sum_{k=0}^{\infty} v_k$. We will recursively construct a sequence of integers $(J_n)_{n\in\omega}$ and a sequence of finite partial permutations $(\pi_n)_{n\in\omega}$, each with domain $\{0, ..., J_n\}$, which satisfy the following seven conditions for each $n \geq 0$:

(I) π_n extends π_{n-1};

(II) $n \in \{\pi_n(0), ..., \pi_n(J_n)\}$;

(III) the definitions of $\pi_n(J_{n-1}+1)$, ..., $\pi_n(J_n)$ depend only on the values of $x(n)$ and $x(n+1)$;

(IV) $\left|\left|\boldsymbol{v} - \sum_{k=0}^{J_n} v_{\pi_n(k)}\right|\right| < \dfrac{1}{x(n+1)+1}$;

(V) $\left|\left|\sum_{k=J_{n-1}+1}^{j} v_{\pi_n(k)}\right|\right| \leq 36 \cdot \dfrac{1}{x(n)+1}$ for every $j \in \{J_{n-1}+1, ..., J_n\}$;

(VI) there exist $i, j \in \{J_{n-1}, ..., J_n\}$ for which $\left|\left|\sum_{k=i}^{j} v_{\pi_n(k)}\right|\right| > \frac{1}{x(n)+1}$; and

(VII) $||v_k|| < \frac{1}{x(n+1)+1} \cdot \frac{1}{C_d}$ for all $k \notin \mathrm{ran}(\pi_n)$.

After this construction is finished, we will let π be the unique permutation which extends all the π_n's, and set $f(x) = \pi$. Conditions (I) and (II) will guarantee that π is indeed a permutation, while (III) will guarantee that the map f is continuous. Conditions (IV) and (V) will ensure that if $x \in \mathcal{C}$, then $\sum_{k=0}^{\infty} v_{\pi(k)}$ will converge to $\boldsymbol{v}$, while condition (VI) will guarantee that if $x \notin \mathcal{C}$, then $\sum_{k=0}^{\infty} v_{\pi(k)}$ will diverge properly. (Condition (VII) is just a technical requirement to facilitate our recursive definition.)

We will now proceed with our construction. Here for the sake of convenience our base case will be $n = -1$. Let $J_{-1} \geq 0$ be so large that $||\boldsymbol{v} - \sum_{k=0}^{J_{-1}} v_k|| < \frac{1}{x(0)+1}$, and that $||v_k|| < \frac{1}{x(0)+1}$ for all $k > J_{-1}$. Let $\pi_{-1} : J_0 + 1 \to J_0 + 1$ be the identity permutation. Note that π_{-1} and J_{-1} trivially satisfy (IV) and (VII) above; this will be enough to facilitate our induction.

Now we assume that J_i and π_i are defined for all $i < n$, and satisfy at least (IV) and (VII), and we proceed with the inductive step of defining J_n and π_n. As we go we will verify that J_n and π_n in fact really do satisfy all of conditions (I)-(VII).

By the Levy-Steinitz theorem, the set of all points $\beta \in \mathbb{R}^n$ for which some rearrangement of $\sum_{k=0}^{\infty} v_k$ converges to β is an affine subspace of $\mathbb{R}^d$; in particular, there is at least a line of possible points to which the rearrangements converge. So we may choose some $\beta \in \mathbb{R}^d$ for which $||\beta - \boldsymbol{v}|| = 3 \cdot \frac{1}{x(n)+1}$, and a permutation $\tau \in S_\infty$ for which $\sum_{k=0}^{\infty} v_{\tau(k)}$ converges to β. Now we will define π_n and J_n by applying the Bounded Walk lemma twice: first, we will use the lemma to "walk out" to a point near β, and then we will use the lemma to "walk back in" to a point near $\boldsymbol{v}$.

To "walk out": apply the Bounded Walk lemma to walk from $\boldsymbol{v}$ to β, extending the finite partial permutation π_{n-1} and with $\rho = \epsilon = \frac{1}{x(n)+1}$. Thus we obtain an index $I > J_{n-1}$ and a finite partial permutation $\sigma : I+1 \to I+1$ which satisfies properties (1)–(5) of the lemma. In particular, condition (3) ensures that we have

$$\begin{aligned}
\left|\left| \sum_{k=J_{n-1}+1}^{I} v_{\sigma(k)} \right|\right| &= \left|\left| \beta - \boldsymbol{v} - \beta + \sum_{k=0}^{I} v_{\sigma(k)} + \boldsymbol{v} - \sum_{k=0}^{J_{n-1}} v_{\sigma(k)} \right|\right| \\
&\geq ||\beta - \boldsymbol{v}|| - \left|\left| \beta - \sum_{k=0}^{I} v_{\sigma(k)} \right|\right| - \left|\left| \boldsymbol{v} - \sum_{k=0}^{J_{n-1}} v_{\sigma(k)} \right|\right| \\
&> 3 \cdot \frac{1}{x(n)+1} - \frac{1}{x(n)+1} - \frac{1}{x(n)+1} \\
&= \frac{1}{x(n)+1}
\end{aligned}$$

while condition (4) guarantees that $\left|\left| \sum_{k=J_{n-1}+1}^{i} v_{\sigma(k)} \right|\right| \leq 6\,||\beta - \boldsymbol{v}|| = 18 \cdot \frac{1}{x(n)+1}$ whenever $J+1 \leq i \leq I$.

Next we "walk back in." Apply the Bounded Walk lemma to walk from β to $\boldsymbol{v}$, extending the finite partial permutation σ, with $\rho = \frac{1}{x(n)+1}$ and $\epsilon = \frac{1}{x(n+1)+1}$. Then we obtain an index $J_n > I$ and a finite partial permutation $\pi_n : J_n + 1 \to J_n + 1$ which again satisfies properties (1)–(5). By the previous inequality, and applying condition (4) for π_n, we see that

$$\begin{aligned}
\left|\left| \sum_{k=J_{n-1}+1}^{i} v_{\pi_n(k)} \right|\right| &\leq \left|\left| \sum_{k=J_{n-1}+1}^{I} v_{\pi_n(k)} \right|\right| + \left|\left| \sum_{k=I+1}^{i} v_{\pi_n(k)} \right|\right| \\
&\leq \left|\left| \sum_{k=J_{n-1}+1}^{I} v_{\sigma(k)} \right|\right| + 6\,||\beta - \boldsymbol{v}|| \\
&\leq 18 \cdot \frac{1}{x(n)+1} + 18 \cdot \frac{1}{x(n)+1} \\
&= 36 \cdot \frac{1}{x(n)+1}
\end{aligned}$$

whenever $I+1 \leq i \leq J_n$. Thus we have shown that (V) holds for π_n. (I), (III), and (IV) obviously hold from our definition of π_n, and (II) holds if we utilize condition (5) in either of our two applications of the Bounded Walk lemma to ensure that $n \in \operatorname{ran}(\pi_n)$. We have shown that (VI) holds if we take $i = J_{n-1}$ and $j = I$, and (VII) follows from condition (2) in our second application of the Bounded Walk lemma. So our construction is complete and we may let $\pi \in S_\infty$ be the unique permutation which extends all of the π_n's.

Define the map $f : \omega^\omega \to S_\infty$ by $f(x) = \pi$, where π is as we have constructed above. The function f, as a map between the Polish space ω^ω and its Polish subspace S_∞, is continuous by condition (III). We claim that f is in fact the continuous reduction we desire.

To see this, suppose $x \in \mathcal{C}$, so $\lim_{n\to\infty} x(n) = \infty$ and hence $\lim_{n\to\infty} \frac{1}{x(n)+1} = 0$. For any $i \in \omega$, let n_i be the greatest integer for which $J_{n_i-1} < i \leq J_{n_i}$. Then by (IV) and (V) we have

$$\begin{aligned}\left\|\boldsymbol{v} - \sum_{k=0}^{i} v_{\pi(k)}\right\| &\leq \left\|\boldsymbol{v} - \sum_{k=0}^{J_{n-1}} v_{\pi_{n-1}(k)}\right\| + \left\|\sum_{k=J_{n-1}+1}^{i} v_{\pi_n(k)}\right\| \\ &\leq \frac{1}{x(n_i)+1} + 36 \cdot \frac{1}{x(n_i)+1} \\ &= 37 \cdot \frac{1}{x(n_i)+1}.\end{aligned}$$

Now taking the limit as $i \to \infty$ (and as $n_i \to \infty$) we see that $\sum_{k=0}^{\infty} v_{\pi(k)}$ converges to $\boldsymbol{v}$. Hence $f(x) \in S_\infty \backslash \mathcal{D}$ and $f(x) \in S_\infty \backslash \mathcal{DP}$.

On the other hand, suppose $x \in \omega^\omega \backslash \mathcal{C}$. Then the sequence $(x(n))$ is cofinally bounded, i.e. there is an $M < \infty$ such that $x(n) \leq M$ infinitely often. Hence $\frac{1}{x(n)+1} > \frac{1}{M+1}$ infinitely often. It follows from (VI) that there are infinitely many blocks $i, ..., j$ of integers for which $\left\|\sum_{k=i}^{j} v_{\pi(k)}\right\| > \frac{1}{x(n)+1} > \frac{1}{M+1}$, and hence $\sum_{k=0}^{\infty} v_{\pi(k)}$ diverges by the Cauchy criterion. In addition, we have already demonstrated that for any n_i depending on i as above, we have

$$\left\|\boldsymbol{v} - \sum_{k=0}^{i} v_{\pi(k)}\right\| \leq 37 \cdot \tfrac{1}{x(n_i)+1} \leq 37.$$

This implies that all partial sums of the rearranged series are bounded, and so the series must in fact diverge properly. Thus in this case we have $f(x) \in \mathcal{D}$ and $f(x) \in \mathcal{DP}$. So f is the reduction we seek, and $\mathcal{D}$ and $\mathcal{DP}$ are $\boldsymbol{\Sigma}^0_3$-complete. □

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