Two Heads Are Better Than None
Or, Me and the Fibonaccis

Steve Kennedy
(with help from Matt Stafford)
Carleton College and
MAA Books

Problems are the lifeblood of mathematics.
— David Hilbert

Indiana MAA Spring Section Meeting 2014
The Problem

The Game:

Flip a fair coin repeatedly until two consecutive heads appear, stop.
The Problem

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Flip a fair coin repeatedly until two consecutive heads appear, stop.

The Problem:

▶ What is the probability that the game will ever end?
(Intuitively, this is big, but consider HTHTHTHTHT..., or TTTTTT....)
The Problem

The Game:

Flip a fair coin repeatedly until two consecutive heads appear, stop.

The Problem(s):

- What is the probability that the game will ever end? (Intuitively, this is big, but consider HTHTHTHTHT..., or TTTTTT...)
- What is the probability that the game ends after exactly \( n \) flips?
Counting Arguments

\[ n = 2 \quad \text{HH} \quad \text{TH} \quad \text{HT} \quad \text{TT} \]

Each event equally likely, so \( P_2 = \frac{1}{4} \).
Counting Arguments

\[ n = 2 \]

\[
\begin{array}{ll}
HH & TH \\
HT & TT
\end{array}
\]

Each event is equally likely, so \( P_2 = 1/4 \).

\[ n = 3 \]

\[
\begin{array}{ll}
HHH & THH \\
HHT & THT \\
HTH & TTH \\
HTT & TTT
\end{array}
\]

Each event is equally likely, so \( P_3 = 1/8 \).
Counting Arguments: Wait just one minute!

\begin{align*}
n = 2 & \quad \text{HH} \quad \text{TH} \\
       & \quad \text{HT} \quad \text{TT} \\
\end{align*}

Each event is equally likely, so $P_2 = 1/4$.

\begin{align*}
n = 3 & \quad \text{HHH} \quad \text{THH} \\
       & \quad \text{HHT} \quad \text{THT} \\
       & \quad \text{HTH} \quad \text{TTT} \\
       & \quad \text{HHT} \quad \text{TTH} \\
       & \quad \text{HTT} \quad \text{TTT} \\
\end{align*}

Each event is equally likely, so $P_3 = 1/8$. 
Yes, neither HHH nor HHT can actually happen. So, the probability of seeing THH is really 1/6. Right?
Neither HHH nor HHT can actually happen. So, the probability of seeing THH is really $1/6$?

Well, not exactly. The probability of seeing THH is $1/6 \times$ the probability that the game didn’t end in exactly two flips.
Non–Counting Arguments

Neither HHH nor HHT can actually happen. So, the probability of seeing THH is really $1/6$?

Well, not exactly. The probability of seeing THH is $1/6 \times \frac{6}{8}$.

So, actually, the probability of seeing THH is:

$$\frac{1}{6} \times \frac{6}{8}.$$
Non–Counting Arguments

Yes, neither HHH nor HHT can actually happen. So, the probability of seeing THH is really 1/6.

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\frac{1}{6} \times \frac{6}{8} = \frac{1}{8}.
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This had to work out.
Non–Counting Arguments

Yes, neither HHH nor HHT can actually happen. So, the probability of seeing THH is really $1/6$.

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So, actually, the probability of seeing THH is:

$$\frac{1}{6} \times \frac{6}{8} = \frac{1}{8}.$$

This had to work out—because the coin doesn’t know the rules of the game.
Counting Arguments

\[ n = 2 \]

HH TH
HT TT

Each event is equally likely, so \( P_2 = 1/4 \).

\[ n = 3 \]

HHH THH
HHT THT
HTH TTH
HTT TTT

Each event is equally likely, so \( P_3 = 1/8 \).

\[ n = 4 \]

HHHH HTHH THHH TTHH
HHHT HTHT THHT TTHT
HHTH HTTH THTH TTTT
HHTT HTTT THTH TTTT

Each event is equally likely, so \( P_4 = 2/16 \).
Counting Arguments

n=5

HHHHH  HTHHH  THHHH  TTHHH
HHHHT  HTHHT  THHHT  TTHHT
HHHTH  HTHTH  THHTH  TTHTH
HHHTT  HTHTT  THHTT  TTHTT
HHTHH  HTTHH  THTHH  TTHHH
HHTHT  HTTHT  THTHT  TTTHT
HHTTH  HTTTH  THTTH  TTTTH
HHTTT  HTTTT  THTTT  TTTTT

So, \( P_5 = \frac{3}{32} \).
Counting Arguments

\begin{enumerate}
\item \text{HHH HTHH THHHH TTHHH HHHHT HTHHT THHHT TTHHT HHHHT HTHHT THHHT TTHHT HHHHT HTHHT THHHT TTHHT}
\item \text{HHHTH HTHTH THHTH TTTTHH HHTHT HTTHT THTHT TTTHT}
\item \text{HHTTH HTTTH THTTH TTTTH}
\end{enumerate}

So, \( P_5 = 3/32 \).

\begin{enumerate}
\item \text{HTHTHH, HTTTHHH, THTTHH, TTHTHH, TTTTHH}
\end{enumerate}

So, \( P_6 = 5/64 \).
Counting Arguments

\[
\begin{align*}
\text{n=5} & & 
\text{HHHHH} & \text{HTHHH} & \text{THHHH} & \text{TTHHH} \\
            & & \text{HHHHT} & \text{HTHHT} & \text{THHHT} & \text{TTHHT} \\
            & & \text{HHHTH} & \text{HTHTH} & \text{THHTH} & \text{TTHTH} \\
            & & \text{HHHTT} & \text{HTHTT} & \text{THHTT} & \text{TTHTT} \\
            & & \text{HHTHH} & \text{HTTTH} & \text{THTTH} & \text{TTTTH} \\
            & & \text{HHTHT} & \text{HTTHT} & \text{THTHT} & \text{TTTHT} \\
            & & \text{HHTTH} & \text{HTTTT} & \text{THTTH} & \text{TTTTT} \\
            & & \text{HHTTT} & \text{HTTTT} & \text{THTTT} & \text{TTTTT} \\
\end{align*}
\]

So, \( P_5 = \frac{3}{32} \).

\[
\begin{align*}
\text{n=6} & & \text{HTHTHH, HTTTHH, THTTHH, TTHTHH, TTTTHH} \\
\end{align*}
\]

So, \( P_6 = \frac{5}{64} \).

\[
\begin{align*}
\text{n=7} \\
\end{align*}
\]
## Counting Arguments

<table>
<thead>
<tr>
<th>n=5</th>
<th>HHHHH</th>
<th>HTHHH</th>
<th>THHHH</th>
<th>TTHHH</th>
</tr>
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<tbody>
<tr>
<td></td>
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<td>HTHHT</td>
<td>THHHT</td>
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<td>HHTTT</td>
<td>HHTTT</td>
<td>THTTT</td>
<td>TTTTT</td>
</tr>
</tbody>
</table>

So, \( P_5 = \frac{3}{32} \).

<table>
<thead>
<tr>
<th>n=6</th>
<th>HTHTHH, HHTTHH, THTTHH, TTHTHH, TTTTHH</th>
</tr>
</thead>
</table>

So, \( P_6 = \frac{5}{64} \).

<table>
<thead>
<tr>
<th>n=7</th>
<th>Homework</th>
</tr>
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</table>

Search for Pattern

\[1, \frac{1}{4}, \frac{1}{8}, \frac{2}{16}, \frac{3}{32}, \frac{5}{64}, \ldots\]
Search for Pattern

\[ \frac{1}{4}, \frac{1}{8}, \frac{2}{16}, \frac{3}{32}, \frac{5}{64}, \ldots \]

\[ P_n = \frac{x_n}{2^n}, \quad \text{What's } x_n? \]
Search for Pattern

\[
\frac{1}{4}, \frac{1}{8}, \frac{2}{16}, \frac{3}{32}, \frac{5}{64}, \ldots
\]

\(P_n = \frac{x_n}{2^n}, \quad \text{What's } x_n?\)

\(x_n = 1, 1, 2, 3, 5, 8, 13, 21, \ldots\)
An Unnecessary(?) Aside

The Fibonacci Numbers — 1202
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The Fibonacci Numbers — 1202

The Rules for Rabbit Reproduction

1. Gestation period is one month.
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3. Rabbits reach sexual maturity in one month.
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The Fibonacci Numbers — 1202

The Rules for Rabbit Reproduction

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   (And never die.)
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The Rules for Rabbit Reproduction

1. Gestation period is one month.
2. Rabbits born in male/female pairs.
3. Rabbits reach sexual maturity in one month.
4. Rabbits are always pregnant.
   (And never die.)

Start with one newborn pair, how many pairs will you have $n$ months later?
The Fibonacci Numbers — 1202

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<tr>
<th>Month:</th>
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<th>2</th>
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Thus, \( F_{n+1} = F_n + F_{n-1} \).
An Answer

**Theorem**

*The sequence $x_n$ is the Fibonacci sequence.*

**Example**

<table>
<thead>
<tr>
<th>$n=6$</th>
<th>$T$</th>
<th>$T$</th>
<th>$HT$</th>
<th>$HT$</th>
<th>$T$</th>
</tr>
</thead>
</table>
An Answer

Theorem

The sequence $x_n$ is the Fibonacci sequence.

Example

<table>
<thead>
<tr>
<th></th>
<th>THTTHH</th>
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</tr>
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<tbody>
<tr>
<td>n=6</td>
<td>TTHTHH</td>
<td>HTTTHH</td>
</tr>
<tr>
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So, $x_6 \leq x_5 + x_4$. 
An Answer

Theorem

The sequence $x_n$ is the Fibonacci sequence.

Example

\begin{align*}
\text{n=6} & \quad \text{THTTTHH} & \text{HTHTTHH} \\
\text{TTHTTHH} & \quad \text{HTTTTHH} \\
\text{TTTTHH} & \quad \text{TTTTHH}
\end{align*}

So, $x_6 \leq x_5 + x_4$.

Conversely

\begin{align*}
\text{*HTTTHH} & \quad \text{**HTTHH} \\
\text{*THTTHH} & \quad \text{**TTHH} \\
\text{*TTTTHH} & \quad \text{TTTTHH}
\end{align*}
An Answer

Theorem
The sequence $x_n$ is the Fibonacci sequence.

Example

$n=6$

So, $x_6 \leq x_5 + x_4$.

Conversely

So, $x_6 \geq x_5 + x_4$. 
More Questions

- What is the probability that the game ends after two or three flips?

- What is the probability that the game ends in four or fewer flips?

- What is the probability that the game ends in twenty or fewer flips?

\[ \sum_{n=2}^{20} P_n \approx 0.983 \]
More Questions

- What is the probability that the game ends after two or three flips?

\[
\frac{1}{4} + \frac{1}{8} = \frac{3}{8}
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- What is the probability that the game ends in four or fewer flips?
  \[
  \frac{1}{4} + \frac{1}{8} + \frac{2}{16} = \frac{1}{2}
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- What is the probability that the game ends in twenty or fewer flips?
  \[ \sum_{n=2}^{20} P_n \approx .983 \]
What is the probability that the game ever ends?
What is the probability that the game ever ends?

\[ \lim_{n \to \infty} \sum_{k=2}^{n} P_n = \sum_{k=0}^{\infty} \frac{F_k}{2^{k+2}} \]
What is the probability that the game ever ends?

\[
\lim_{n \to \infty} \sum_{k=2}^{n} P_n = \sum_{k=0}^{\infty} \frac{F_k}{2^{k+2}}
\]

Does this series converge?
Convergence

\[
\frac{1}{4} + \frac{1}{8} + \frac{2}{16} + \frac{3}{32} + \frac{5}{64} + \frac{8}{128} + \ldots = S
\]
Convergence

\[ \frac{1}{4} + \frac{1}{8} + \frac{2}{16} + \frac{3}{32} + \frac{5}{64} + \frac{8}{128} + \ldots = S \]

\[ \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64} + \frac{1}{128} + \ldots = \frac{1}{2} \]
Convergence

\[
\frac{1}{4} + \frac{1}{8} + \frac{3}{16} + \frac{5}{32} + \frac{8}{64} + \frac{16}{128} + \ldots = S
\]

\[
\frac{1}{16} + \frac{1}{32} + \frac{1}{64} + \frac{1}{128} + \ldots = 1
\]

\[
\frac{1}{64} + \frac{1}{128} + \ldots = \frac{1}{8}
\]
Convergence

\[
\frac{1}{4} + \frac{1}{8} + \frac{2}{16} + \frac{3}{32} + \frac{5}{64} + \frac{8}{128} + \ldots = S
\]

\[
1 + 1 + \frac{1}{16} + \frac{1}{32} + \frac{1}{64} + \frac{1}{128} + \ldots = \frac{1}{2}
\]

\[
\frac{1}{16} + \frac{1}{32} + \frac{1}{64} + \frac{1}{128} + \ldots = \frac{1}{8}
\]

\[
\frac{1}{32} + \frac{1}{64} + \frac{1}{128} + \ldots = \frac{1}{16}
\]
Convergence

\[
\frac{1}{4} + \frac{1}{8} + \frac{2}{16} + \frac{3}{32} + \frac{5}{64} + \frac{8}{128} + \ldots = S
\]  
\[
\frac{1}{4} + \frac{1}{8} + \frac{1}{16} \quad + \quad \frac{1}{32} + \frac{1}{64} + \frac{1}{128} + \ldots = \frac{1}{2}
\]  
\[
\frac{1}{16} + \frac{1}{32} + \frac{1}{64} + \frac{1}{128} + \ldots = \frac{1}{8}
\]  
\[
\frac{1}{32} + \frac{1}{64} + \frac{1}{128} + \ldots = \frac{1}{16}
\]  
\[
\frac{1}{64} + \frac{1}{128} + \ldots = \frac{1}{32}
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Convergence

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\frac{1}{4} + \frac{1}{8} + \frac{2}{16} + \frac{3}{32} + \frac{5}{64} + \frac{8}{128} + \ldots = S
\]

\[
\frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64} + \frac{1}{128} + \ldots = 1
\]

\[
\frac{1}{16} + \frac{1}{32} + \frac{1}{64} + \frac{1}{128} + \ldots = 1
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\]

\[
\frac{1}{64} + \frac{1}{128} + \ldots = \frac{1}{32}
\]

\[
\frac{1}{128} + \ldots = \frac{1}{64}
\]

\[
\ldots
\]

\[
\frac{1}{128} + \ldots = \frac{1}{64}
\]

\[
\ldots
\]
Convergence

\[
\begin{align*}
\frac{1}{4} + \frac{1}{8} + \frac{2}{16} + \frac{3}{32} + \frac{5}{64} + \frac{8}{128} + \ldots &= S \\
\frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64} + \frac{1}{128} + \ldots &= \frac{1}{2} \\
\frac{1}{16} + \frac{1}{32} + \frac{1}{64} + \frac{1}{128} + \ldots &= \frac{1}{8} \\
\frac{1}{32} + \frac{1}{64} + \frac{1}{128} + \ldots &= \frac{1}{16} \\
\frac{1}{64} + \frac{1}{128} + \ldots &= \frac{1}{32} \\
\frac{1}{128} + \ldots &= \frac{1}{64} \\
\end{align*}
\]

So, \( S = \frac{1}{2} + \frac{1}{8} + \frac{1}{16} + \frac{2}{32} + \frac{3}{64} + \ldots \)
The Sum

Recall,

\[ S = \frac{1}{4} + \frac{1}{8} + \frac{2}{16} + \frac{3}{32} + \frac{5}{64} + \frac{8}{128} + \ldots \]
The Sum

Recall,

\[ S = \frac{1}{4} + \frac{1}{8} + \frac{2}{16} + \frac{3}{32} + \frac{5}{64} + \frac{8}{128} + \ldots \]

We just decided,

\[ S = \frac{1}{2} + \frac{1}{8} + \frac{1}{16} + \frac{2}{32} + \frac{3}{64} + \frac{5}{128} + \ldots \]
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We just decided,

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So,

\[ S = \frac{1}{2} + \frac{1}{2} \left( \frac{1}{4} + \frac{1}{8} + \frac{2}{16} + \frac{3}{32} + \frac{5}{64} + \ldots \right) \]

\[ S = \frac{1}{2} + \frac{1}{2} S \]
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\[ S = \frac{1}{2} + \frac{1}{8} + \frac{1}{16} + \frac{2}{32} + \frac{3}{64} + \frac{5}{128} + \ldots \]

So, 

\[ S = \frac{1}{2} + \frac{1}{2} \left( \frac{1}{4} + \frac{1}{8} + \frac{2}{16} + \frac{3}{32} + \frac{5}{64} + \ldots \right) \]

\[ S = \frac{1}{2} + \frac{1}{2} S \]

\[ S = 1 \]
The Sum

\[ \frac{1}{4} + \frac{1}{8} + \frac{2}{16} + \frac{3}{32} + \frac{5}{64} + \ldots = 1 \]

\[ \sum_{n=2}^{\infty} \frac{F_{n-2}}{2^n} = 1 \]
The Sum

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The Sum

\[ \sum_{n=2}^{\infty} \frac{F_{n-2}}{2^n} = 1!!! \]
The Miracle

\[ \sum_{n=2}^{\infty} \frac{F_{n-2}}{2^n} = 1!!! \]
The Miracle

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THIS IS A MIRACLE!!!
The Miracle

Definition

A *miracle* is the simultaneous occurrence of two or more zero-probability events.
The Miracle

Definition
A miracle is the simultaneous occurrence of two or more zero-probability events.

- The series converged.
The Miracle

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A *miracle* is the simultaneous occurrence of two or more zero-probability events.

► The series converged. (And I could prove it!)
The Miracle

Definition
A *miracle* is the simultaneous occurrence of two or more zero-probability events.

- The series converged. (And I could prove it!)
- We could find the value to which it converged.
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A *miracle* is the simultaneous occurrence of two or more zero-probability events.

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- We could find the value to which it converged.
- That value was rational.
Definition
A *miracle* is the simultaneous occurrence of two or more zero-probability events.

- The series converged. (And I could prove it!)
- We could find the value to which it converged.
- That value was rational.
- That value was the simplest possible rational.
"I think you should be more explicit here in step two."
The Very Good Reason

\[ g(x) = \sum_{k=0}^{\infty} F_k x^k = 1 + x + 2x^2 + 3x^3 + 5x^4 + \ldots \]
The Very Good Reason

\[ g(x) = \sum_{k=0}^{\infty} F_k x^k = 1 + x + 2x^2 + 3x^3 + 5x^4 + \ldots \]

\[ x \cdot g(x) = \sum_{k=0}^{\infty} F_k x^{k+1} = x + x^2 + 2x^3 + 3x^4 + \ldots \]
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The Very Good Reason

\[ g(x) = \sum_{k=0}^{\infty} F_k x^k = 1 + x + 2x^2 + 3x^3 + 5x^4 + \ldots \]

\[ xg(x) = \sum_{k=0}^{\infty} F_k x^{k+1} = x + x^2 + 2x^3 + 3x^4 + \ldots \]

\[ x^2g(x) = \sum_{k=0}^{\infty} F_k x^{k+2} = x^2 + x^3 + 2x^4 + \ldots \]

So,

\[ g(x) - xg(x) - x^2g(x) = 1 \]
The Very Good Reason

\[ g(x) = \sum_{k=0}^{\infty} F_k x^k = 1 + x + 2x^2 + 3x^3 + 5x^4 + \ldots \]

\[ x g(x) = \sum_{k=0}^{\infty} F_k x^{k+1} = x + x^2 + 2x^3 + 3x^4 + \ldots \]

\[ x^2 g(x) = \sum_{k=0}^{\infty} F_k x^{k+2} = x^2 + x^3 + 2x^4 + \ldots \]

So,

\[ g(x) - xg(x) - x^2 g(x) = 1 \]

\[ g(x) = \frac{1}{1 - x - x^2} \]
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\[ g \left( \frac{1}{2} \right) = \sum_{k=0}^{\infty} F_k \left( \frac{1}{2} \right)^k = 1 + \frac{1}{2} + 2 \left( \frac{1}{2} \right)^2 + 3 \left( \frac{1}{2} \right)^3 + 5 \left( \frac{1}{2} \right)^4 + \ldots \]
The Very Good Reason

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\[ = 1 + \frac{1}{2} + \frac{2}{4} + \frac{3}{8} + \frac{5}{16} + \ldots \]
The Very Good Reason

\[ g(x) = \frac{1}{1 - x - x^2} \]

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\[ = 1 + \frac{1}{2} + \frac{2}{4} + \frac{3}{8} + \frac{5}{16} + \ldots \]

\[ = 4 \left( \frac{1}{4} + \frac{1}{8} + \frac{2}{16} + \frac{3}{32} + \frac{5}{64} + \ldots \right) \]
Technicalities

\[ \sum_{k=0}^{\infty} F_k x^k = 1 + x + 2x^2 + 3x^3 + 5x^4 + \ldots \]

This is the Taylor series about 0 for \( \frac{1}{1-x-x^2} \).

The interval of convergence is \((-\frac{1}{\lambda}, \frac{1}{\lambda})\), where \( \lambda = 1 + \sqrt{\frac{5}{2}} \), i.e. the golden mean.

(NB The radius of convergence, \( \frac{1}{\lambda} \), is approximately \( 0.618 \), so \( \frac{1}{2} \) is comfortably inside.)
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\[
\sum_{k=0}^{\infty} F_k x^k = 1 + x + 2x^2 + 3x^3 + 5x^4 + \ldots
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(NB The radius of convergence, \( \frac{1}{\lambda} \), is approximately .618, so \( \frac{1}{2} \) is comfortably inside.)
Two Great Tastes That Taste Great Together

The function, \( g(x) = \frac{1}{1-x-x^2} \), is called the generating function for the Fibonacci numbers.
Rewrite $g(x)$ using partial fractions:

\[
\frac{1}{1 - x - x^2} = \frac{A}{1 - \lambda x} + \frac{B}{1 + (\lambda - 1)x}
\]
Rewrite \( g(x) \) using partial fractions:

\[
\frac{1}{1 - x - x^2} = \frac{A}{1 - \lambda x} + \frac{B}{1 + (\lambda - 1)x}
\]

Solve for \( A \) and \( B \):

\[
A = \frac{\lambda}{\sqrt{5}} \quad B = \frac{\lambda - 1}{\sqrt{5}}
\]
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\[
\frac{1}{1 - x - x^2} = \frac{A}{1 - \lambda x} + \frac{B}{1 + (\lambda - 1)x}
\]

Solve for \( A \) and \( B \):

\[
A = \frac{\lambda}{\sqrt{5}} \quad B = \frac{\lambda - 1}{\sqrt{5}}
\]

By the known formula for geometric series:

\[
\frac{A}{1 - \lambda x} = \sum_{k=0}^{\infty} A (\lambda x)^k
\]
Rewrite \( g(x) \) using partial fractions:

\[
\frac{1}{1 - x - x^2} = \frac{A}{1 - \lambda x} + \frac{B}{1 + (\lambda - 1)x}
\]

Solve for \( A \) and \( B \):

\[
A = \frac{\lambda}{\sqrt{5}} \quad \quad \quad \quad B = \frac{\lambda - 1}{\sqrt{5}}
\]

By the known formula for geometric series:

\[
\frac{A}{1 - \lambda x} = \sum_{k=0}^{\infty} A(\lambda x)^k
\]

Similarly,

\[
\frac{B}{1 + (\lambda - 1)x} = \sum_{k=0}^{\infty} B(-1)^k ((\lambda - 1)x)^k
\]
Never in a Million Years

\[ \frac{1}{1 - x - x^2} = \sum_{k=0}^{\infty} A(\lambda x)^k + \sum_{k=0}^{\infty} B(-1)^k ((\lambda - 1)x)^k \]
Never in a Million Years

\[
\frac{1}{1 - x - x^2} = \sum_{k=0}^{\infty} A(\lambda x)^k + \sum_{k=0}^{\infty} B(-1)^k ((\lambda - 1)x)^k
\]

Recalling the values of \(A\) and \(B\),

\[
\sum_{k=0}^{\infty} F_k x^k = \sum_{k=0}^{\infty} \left[ \frac{\lambda^{k+1}}{\sqrt{5}} + \frac{(-1)^k (\lambda - 1)^{k+1}}{\sqrt{5}} \right] x^k
\]
\[
\frac{1}{1 - x - x^2} = \sum_{k=0}^{\infty} A(\lambda x)^k + \sum_{k=0}^{\infty} B(-1)^k ((\lambda - 1)x)^k
\]

Recalling the values of \(A\) and \(B\),

\[
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\]

Power series expansions are unique! So,

\[
F_k = \frac{\lambda^{k+1}}{\sqrt{5}} + \frac{(-1)^k(\lambda - 1)^{k+1}}{\sqrt{5}}
\]
Be wise, generalize!

<table>
<thead>
<tr>
<th>Heads</th>
<th>Numerators and Recursion</th>
<th>Generating Function</th>
</tr>
</thead>
<tbody>
<tr>
<td>One</td>
<td>1, 1, 1, 1, ...</td>
<td>( \frac{1}{1-x} )</td>
</tr>
<tr>
<td>Two</td>
<td>1, 1, 2, 3, ...</td>
<td>( \frac{1}{1-x-x^2} )</td>
</tr>
<tr>
<td>Three</td>
<td>1, 1, 2, 4, 7, ...</td>
<td>( \frac{1}{1-x-x^2-x^3} )</td>
</tr>
<tr>
<td>Four</td>
<td>1, 1, 2, 4, 8, 15, ...</td>
<td>( \frac{1}{1-x-x^2-x^3-x^4} )</td>
</tr>
</tbody>
</table>

And so on ...
"Suppose you had a three-sided coin?"

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</thead>
<tbody>
<tr>
<td>One</td>
<td>1, 2, 4, 8, \ldots</td>
<td>\frac{1}{1-2x}</td>
</tr>
<tr>
<td></td>
<td>(a_n = 2a_{n-1})</td>
<td></td>
</tr>
<tr>
<td>Two</td>
<td>1, 2, 6, 16, 48, \ldots</td>
<td>\frac{1}{1-2x-2x^2}</td>
</tr>
<tr>
<td></td>
<td>(a_n = 2(a_{n-1} + a_{n-2}))</td>
<td></td>
</tr>
<tr>
<td>Three</td>
<td>1, 2, 6, 18, 52, 152, \ldots</td>
<td>\frac{1}{1-2x-2x^2-2x^3}</td>
</tr>
<tr>
<td></td>
<td>(a_n = 2(a_{n-1} + a_{n-2} + a_{n-3}))</td>
<td></td>
</tr>
<tr>
<td>Four</td>
<td>1, 2, 6, 18, 54, 160, \ldots</td>
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<td>(a_n = 2(a_{n-1} + a_{n-2} + a_{n-3} + a_{n-4}))</td>
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</tr>
</tbody>
</table>

Homework: Do the \(n\) consecutive heads from an \(m\)-sided coin problem.
Fibonacci Fractions

Do the long division problem:

\[
\frac{10,000}{9,899} =
\]
Fibonacci Fractions

Do the long division problem:

\[
\frac{10,000}{9,899} = 1.010203050813213455 \ldots
\]
Fibonacci Fractions

Do the long division problem:

\[
\frac{10,000}{9,899} = 1.010203050813213455 \ldots
\]

What just happened?

\[
g \left( \frac{1}{100} \right) = 1 + \frac{1}{100} + \frac{2}{(100)^2} + \frac{3}{(100)^3} + \ldots
\]

\[
= 1 + .01 + .0002 + .000003 + \ldots
\]
Fibonacci Fractions

Do the long division problem:

\[
\frac{10,000}{9,899} = 1.010203050813213455\ldots
\]

What just happened?

\[
g \left( \frac{1}{100} \right) = 1 + \frac{1}{100} + \frac{2}{(100)^2} + \frac{3}{(100)^3} + \ldots
\]

\[
= 1 + .01 + .0002 + .000003 + \ldots
\]

\[
g \left( \frac{1}{1000} \right) = \frac{1,000,000}{998,999}
\]
Recall

\[ F_k = \frac{1}{\sqrt{5}} \left[ \lambda^{k+1} + (-1)^k (\lambda - 1)^{k+1} \right] \]
Multiplying Weirdness

Recall

\[ F_k = \frac{1}{\sqrt{5}} \left[ \lambda^{k+1} + (-1)^k (\lambda - 1)^{k+1} \right] \]

Here’s a picture of those fractional parts:

Pictured is the fractional part of \( \frac{1}{\sqrt{5}} \lambda^k \), \( k = 1, 2, \ldots, 15 \).
Your Real Homework

The picture for the fractional part of 1.5\(^n\) for \(n = 10 \ldots 80\). (This is what we expect to see.)
Conjecture

There are not very many real numbers, $\gamma$, that have the property that there exists a constant $C$ so that the sequence consisting of the fractional parts of $C\gamma^n$, $n = 1, 2, \ldots$ has only finitely many limit points.
Conjecture

There are not very many real numbers, $\gamma$, that have the property that there exists a constant $C$ so that the sequence consisting of the fractional parts of $C\gamma^n$, $n = 1, 2, \ldots$ has only finitely many limit points.

Not very many $=$ measure zero
   $=$ first category
   $=$ nowhere dense
The number $1 + \sqrt{2} = p$ is even better than the golden mean.

\[
\begin{align*}
    p^1 & \approx 2.41421 \\
    p^2 & \approx 5.82842 \\
    p^3 & \approx 14.07106 \\
    \ldots \\
    p^8 & \approx 1153.99913 \\
    p^9 & \approx 2786.00035 \\
    p^{10} & \approx 6725.99985
\end{align*}
\]
Your Real Homework: Hints

\[ p^1 = 1 + \sqrt{2} \]
\[ p^2 = 3 + 2\sqrt{2} \]
\[ p^3 = 7 + 5\sqrt{2} \]
\[ \ldots \]
\[ p^8 = 577 + 408\sqrt{2} \]
\[ p^9 = 1393 + 985\sqrt{2} \]
\[ p^{10} = 3363 + 2378\sqrt{2} \]
A Final Word

Thanks to the local organizers: Adam Coffman, Rob Merkovsky and Marc Lipman.
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Thanks to IUPU–Fort Waye.
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Thank you for your kind attention.