Galois groups of rational functions with non-trivial automorphisms

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Galois representations from elliptic curves

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In fact, for all but finitely many p, G_∞ injects into a Cartan subgroup C.

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Answer: (Serre 1972) If E has no CM, then $[\mathrm{GL}_2(\mathbb{Z}_p) : G_\infty] < \infty$. If E has CM and $G_\infty \hookrightarrow C$, then $[C : G_\infty] < \infty$. Moreover, in either case for all but finitely many p the index is 1.

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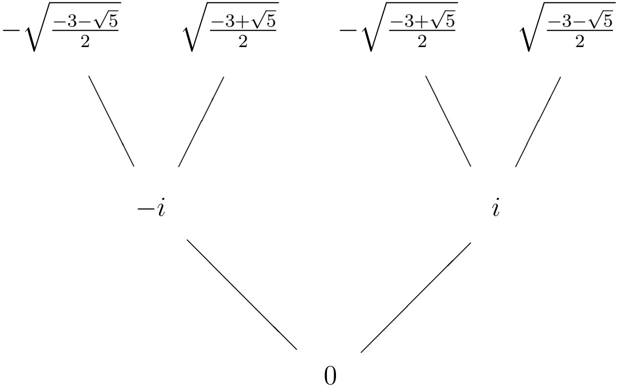
Unlike the elliptic curves case, $\phi^{-n}(0)$ has no group structure, but $T_0 := \bigcup_n \phi^{-n}(0)$ has a natural tree structure. So $G_\infty \hookrightarrow \mathrm{Aut}(T_0)$.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First two levels of preimage tree T_0 for $\phi(x) = \frac{x^2+1}{x}$, initial point 0.

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Even restricting to quadratic polynomials, there are very few results. It is known that $[\mathrm{Aut}(T_0) : G_\infty] < \infty$ for

- $\phi(x) = x^2 + a$, for $a > 0, a \equiv 1, 2 \bmod 4$ and $a < 0, a \equiv 0 \bmod 4$ (Stoll 1992),
- $\phi(x) = x^2 - ax + a$, $a \in \mathbb{Z}$
  $\phi(x) = x^2 + ax - 1, a \in \mathbb{Z} \setminus \{0, 2\}$ (RJ 2008).

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Let $\mathrm{Aut}(\phi)$ be the group of Mobius transformations commuting with ϕ.

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Proposition

Let (ϕ, b) consist of a quadratic rational function ϕ and basepoint $b \in \mathbb{P}^1(\mathbb{Q})$ such that ϕ commutes with a Mobius transformation α of order 2 and $\alpha(b) = b$. Then (ϕ, b) is conjugate to

$$\left(\frac{k(x^2+m)}{cx}, 0\right)$$

for some $k, m, c \in \mathbb{Z}$.

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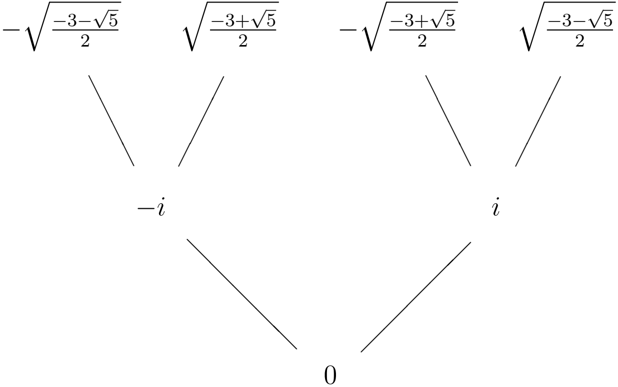
Remark

Let $T_{0,n}$ be the truncation of T_0 to the first n levels only, and define $C_n(\alpha)$ similarly. Then $C_n(\alpha)$ contains a subgroup of index two that is isomorphic to $\mathrm{Aut}(T_{0,n-1})$.



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$$C_2(\alpha) = \{e, (1\ \ 2)(3\ \ 4), (1\ \ 3)(2\ \ 4), (1\ \ 4)(2\ \ 3)\}$$

Theorems for quadratic rational functions

Theorem (RJ-Michelle Manes)

Let $\phi = \frac{k(x^2+1)}{x}$ for $k \in \mathbb{Z}$, and define $p_n(x)$ to be the numerator of $\phi^n(x)$. Suppose that for all $n \geq 2$, $kp_n(1)$ is not a square in $\mathbb{Z}$. Then $[C(\alpha) : G_\infty] < \infty$.

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Example

If $k = 1$, then $p_n(1)$ is the left coordinate in the recurrence given by $(r_0, s_0) = (1, 1)$, $(r_n, s_n) = (r_{n-1}^2 + s_{n-1}^2, r_{n-1}s_{n-1})$, which proceeds

$$(1, 1), (2, 1), (5, 2), (29, 10), (941, 290), \ldots$$

One can check that in the above example, $p_n(1) \equiv 2 \bmod 3$ for all $n \geq 2$, so the Theorem applies.

Corollary

Suppose that $\phi = \frac{k(x^2+1)}{x}$ *and* $k \bmod 24 \notin \{2, 6, 8, 12, 14, 18, 20\}$*. Then* $[C(\alpha) : G_\infty] < \infty$*.*

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Corollary

Suppose that $\phi = \frac{k(x^2+1)}{x}$ *and* $k \bmod 24 \notin \{2, 6, 8, 12, 14, 18, 20\}$. *Then* $[C(\alpha) : G_\infty] < \infty$.

Proof: Find p such that for k satisfying certain congruences $\bmod p$, $p_n(1)$ is a fixed non-square $\bmod p$ for all $n \geq 2$.

Theorem (RJ-Michelle Manes)

Let $\phi = \frac{k(x^2+1)}{x}$ *for* $k \in \mathbb{Z}$*, and suppose that* $p_n(1)$ *is not a square for all* $n \geq 2$*. Let* v_p *denote the* p*-adic valuation, and assume in addition that* $v_p(k) = 0$ *for all primes* p *dividing some* $p_j(1)$ *for* $\psi = (x^2+1)/x$*. Then* $G_\infty \cong C(\alpha)$*.*

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Remark: Recall that $p_j(1)$ for $\psi = (x^2+1)/x$ is given by the left coordinate in the recurrence $(r_0, s_0) = (1, 1)$, $(r_n, s_n) = (r_{n-1}^2 + s_{n-1}^2, r_{n-1}s_{n-1})$. Thus a prime dividing some $p_j(1)$ must be the sum of two squares, and therefore is 1 mod 4.

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Moreover, one can show the natural density of the set of primes dividing some $p_j(1)$ is zero.

Proof strategy:

1. Show that if there exists a prime $p \in \mathbb{Z}$ that ramifies in $K_n := K(\phi^{-n}(0))$ but not in $K_{n-1} := K(\phi^{-(n-1)}(0))$, then $\mathrm{Gal}(K_n/K_{n-1}) \cong (\ker C_n(\alpha) \to C_{n-1}(\alpha))$.

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4. Assume that $v_q(k) = 0$ for all q dividing $p_j(1)$ for $\psi = (x^2 + 1)/x$. Show that in this case $kp_n(1)$ is divisible to an odd power by a prime not dividing k, for all n.