Potential multiparticle entanglement measure

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In this Brief Report we discuss entanglement of multiparticle quantum systems. We propose a potential measure of a type of entanglement of pure states of n qubits, the n-tangle. For a system of two qubits the n-tangle is equal to the square of the concurrence, and for systems of three qubits it is equal to the ''residual entanglement.'' We show that the n -tangle is also equal to a generalization of the concurrence squared for even n, and use this fact to prove that the n-tangle is an entanglement monotone. However, the n-tangle is undefined for odd $n>3$. Finally, we propose a measure related to the n-tangle for mixed-state systems of n qubits, and find an analytical formula for this measure for even n.

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I. INTRODUCTION

The quantum phenomenon of entanglement is presently the subject of much active research and discussion. This comes from the fundamental interest in quantum phenomena, and is also due to recent proposals for quantum computation [1,2]. Entanglement is the property that provides a quantum computer with advantages over its classical counterpart. If one is designing a quantum computer, then quantifying the entanglement of a large number of qubits is likely to be valuable. Quantum entanglement allows correlations between separated quantum particles that are not possible in classical systems [3]. Hence entanglement measures should also prove valuable in the quantum applications of cloning, communication, and encryption.

A method for classifying and quantifying the entanglement in a particular state would greatly increase our understanding of this phenomenon: There have been numerous studies of quantum entanglement, with equally numerous entanglement measures proposed [4–14]. There remain many open questions regarding the quantification of entanglement. In particular, states with more than one subsystem have only just begun to be considered. While entanglement measures of pure states are essential, so is their applicability to mixed states. The presence of noise in a quantum channel [15], or the decoherence effects of qubits interacting with an environment [16], will transform an idealized pure state into a mixed one.

One type of multipartide entanglement is n-way or n-party entanglement, entanglement that critically involves all n particles. For example, a three-qubit state with only three-way (or three-party) entanglement has the property that tracing out one of the qubits leaves the other two particles unentangled [7]. It was recently proven that states with n-way entanglement ($n>2$) cannot be reversibly distilled from two-way entanglement [9]. An example of a state with only three-way entanglement is the Greenberger-Horne-Zeilinger (GHZ) state: $|\mathrm{GHZ}\rangle=(|000\rangle+|111\rangle)/\sqrt{2}$, for which case $\tau_{ABC}(|\mathrm{GHZ}\rangle)=1$. The W state, $|W\rangle=(|001\rangle+|010\rangle+|100\rangle)/\sqrt{3}$, with $\tau_{ABC}(|W\rangle)=0$, is an example of a state with two-way entanglement but no three-way entanglement: tracing out one of the particles leaves a partially entangled pair of qubits. In general, three-qubit states have both kinds of entanglement.

The *concurrence* has been shown to be a useful entanglement measure for pure and mixed states with two qubits, and can be related to the *entanglement of formation* [5]. A recent paper by Coffman, Kundu, and Wootters [4] using concurrence to examine three-qubit quantum systems, introduced the concept of ''residual entanglement,'' or the 3-tangle, τ_{ABC}. $\tau_{ABC}(|\psi\rangle)$ is a potential way to quantify the amount of three-way entanglement in system ABC.

In this Brief Report we will show that a generalization of the 3-tangle for n qubits, the n-tangle τ, is related to a generalization of pure-state concurrence for states with an even number of qubits. This allows us to prove that the n-tangle is an entanglement monotone for states with three or an even number of qubits. We also show that the n-tangle is equal to 1 for an n-qubit generalization of the GHZ state [17], and 0 for an n-qubit generalization of the W state [7]. Finally, we introduce a mixed-state measure of entanglement related to the n-tangle that is analogous to the entanglement of formation, and find an analytical formula for this measure for states with an even number of qubits.

The Brief Report is organized as follows. In Sec. II we define the n-tangle, and show that for states with even n, $\tau_{1\ldots n}$ is equal to the square of a natural generalization of pure-state concurrence. Since two-qubit concurrence is related to entanglement and entanglement of formation [5], this suggests that the n-tangle may have a physical interpretation. We prove that $\tau_{1\ldots n}$ is an entanglement monotone [6], which gives further evidence that the n-tangle measures a type of entanglement. We also consider the value of the n-tangle for generalizations of the GHZ and W states and another example state. The extension of our pure-state results to mixed-states is shown in Sec. III. A mixed-state version of the n-tangle, τ^{min}, is introduced, and an analytical formula for $\tau^{min}_{1\ldots n}$ for even n is presented. In Sec. IV we conclude with a discussion of our results.

II. n-TANGLE

For three qubits the ''residual entanglement,'' or τ_{ABC}, is given by

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$$\tau_{ABC}(|\psi\rangle)=2\left|\sum a_{\alpha_1\alpha_2\alpha_3}a_{\beta_1\beta_2\beta_3}a_{\gamma_1\gamma_2\gamma_3}a_{\delta_1\delta_2\delta_3}\right.$$
$$\left.\times\epsilon_{\alpha_1\beta_1}\epsilon_{\alpha_2\beta_2}\epsilon_{\gamma_1\delta_1}\epsilon_{\gamma_2\delta_2}\epsilon_{\alpha_3\gamma_3}\epsilon_{\beta_3\delta_3}\right|, \quad (1)$$

where the a terms are the coefficients in the standard basis defined by $|\psi\rangle=\Sigma_{i_1\ldots i_n}a_{i_1\ldots i_n}|i_1i_2\ldots i_n\rangle$, and $\epsilon_{01}=-\epsilon_{10}=1$ and $\epsilon_{00}=-\epsilon_{11}=0$ [4]. We define the n-tangle by

$$\tau_{1\ldots n}=2\left|\sum a_{\alpha_1\ldots\alpha_n}a_{\beta_1\ldots\beta_n}a_{\gamma_1\ldots\gamma_n}a_{\delta_1\ldots\delta_n}\right.$$
$$\times\epsilon_{\alpha_1\beta_1}\epsilon_{\alpha_2\beta_2}\ldots\epsilon_{\alpha_{n-1}\beta_{n-1}}\epsilon_{\gamma_1\delta_1}\epsilon_{\gamma_2\delta_2}\ldots$$
$$\left.\times\epsilon_{\gamma_{n-1}\delta_{n-1}}\epsilon_{\alpha_n\gamma_n}\epsilon_{\beta_n\delta_n}\right| \quad (2)$$

for all even n and $n=3$. By reasoning similar to that used for $n=3$ [7], the n-tangle is invariant under local unitarities. We show below that the n-tangle is invariant under permutations of the qubits. However, the above formula is *not* invariant under permutations of qubits for general odd n over 3, and hence is not a viable measure of odd-way entanglement (aside from $n=3$).

There is a relationship that can be shown between τ and pure-state concurrence. Pure-state concurrence was defined for states of two qubits in Ref. [5] by $C(\psi)=|\langle\psi|\tilde{\psi}\rangle|^2$, where $|\tilde{\psi}\rangle=\sigma_y^{\otimes n}|\psi^*\rangle$ is the "spin flip" of $|\psi\rangle$ in terms of the Pauli spin matrix $\sigma_y=\begin{pmatrix}0 & -1\\ i & 0\end{pmatrix}$. C is defined only for states of two qubits, but the obvious generalization uses the same equation, $C_{1\ldots n}(\psi)=|\langle\psi|\tilde{\psi}\rangle|^2$, where $|\tilde{\psi}\rangle$ now stands for an n-qubit state. Note that for the two-qubit case, $\tau_{12}=C^2$. We will prove that the analogous equation, $\tau_{1\ldots n}=C^2_{1\ldots n}$, is true for all even n.

We shall find an expression for $C^2_{1\ldots n}$ in terms of the coefficients in the standard basis. One can express an n-qubit state $|\psi\rangle$ as a vector in the standard basis indexed by $|\psi\rangle_{i_1\ldots i_n}$, where each i indexes one of the qubits. Then $|\psi\rangle_{i_1\ldots i_n}=a_{i_1\ldots i_n}$.

Note that $\sigma^{\otimes n}_{y_{i_1\ldots i_n,j_1\ldots j_n}}=\epsilon_{i_1j_1}\ldots\epsilon_{i_nj_n}e^{i\theta}$ for some real θ because $\sigma_{y_{i,j}}=-i\epsilon_{ij}$. Therefore, $|\tilde{\psi}\rangle=\sigma_y^{\otimes n}|\psi^*\rangle$ implies $|\tilde{\psi}\rangle_{i_1\ldots i_n}=\Sigma^1_{\beta_1\ldots\beta_n}a^*_{\beta_1\ldots\beta_n}\epsilon_{i_1\beta_1}\epsilon_{i_2\beta_2}\ldots\epsilon_{i_n\beta_n}e^{i\theta}$, so $\langle\psi|\tilde{\psi}\rangle=\Sigma_{all\ \alpha,\beta}a^*_{\alpha_1\ldots\alpha_n}a^*_{\beta_1\ldots\beta_n}\epsilon_{\alpha_1\beta_1}\epsilon_{\alpha_2\beta_2}\ldots\epsilon_{\alpha_n\beta_n}e^{i\theta}$. Thus,

$$|\langle\psi|\tilde{\psi}\rangle|^2=\left|\sum a_{\alpha_1\ldots\alpha_n}a_{\beta_1\ldots\beta_n}a_{\gamma_1\ldots\gamma_n}a_{\delta_1\ldots\delta_n}\epsilon_{\alpha_1\beta_1}\epsilon_{\alpha_2\beta_2}\ldots\epsilon_{\alpha_n\beta_n}\epsilon_{\gamma_1\delta_1}\epsilon_{\gamma_2\delta_2}\ldots\epsilon_{\gamma_n\delta_n}\right|, \quad (3)$$

where the sum is over all indices. Expanding the last index of each a and using the fact that $\epsilon_{i,j}=-\epsilon_{j,i}$ for even n, one obtains

$$|\langle\psi|\tilde{\psi}\rangle|^2=2\left|-\sum a_{\alpha_1\ldots\alpha_{n-1}0}\ a_{\beta_1\ldots\beta_{n-1}1}\ a_{\gamma_1\ldots\gamma_{n-1}1}\ a_{\delta_1\ldots\delta_{n-1}0}\epsilon_{\alpha_1\beta_1}\epsilon_{\alpha_2\beta_2}\ldots\epsilon_{\alpha_{n-1}\beta_{n-1}}\epsilon_{\gamma_1\delta_1}\epsilon_{\gamma_2\delta_2}\ldots\epsilon_{\gamma_{n-1}\delta_{n-1}}\right.$$
$$\left.-\sum a_{\alpha_1\ldots\alpha_{n-1}1}\ a_{\beta_1\ldots\beta_{n-1}0}\ a_{\gamma_1\ldots\gamma_{n-1}0}\ a_{\delta_1\ldots\delta_{n-1}1}\epsilon_{\alpha_1\beta_1}\epsilon_{\alpha_2\beta_2}\ldots\epsilon_{\alpha_{n-1}\beta_{n-1}}\epsilon_{\gamma_1\delta_1}\epsilon_{\gamma_2\delta_2}\ldots\epsilon_{\gamma_{n-1}\delta_{n-1}}\right|. \quad (4)$$

Now we turn our attention to the expression for τ. Equation (2) can be expanded to

$$\tau_{1\ldots n}=\left|\sum a_{\alpha_1\ldots\alpha_{n-1}0}\ a_{\beta_1\ldots\beta_{n-1}0}\ a_{\gamma_1\ldots\gamma_{n-1}1}\ a_{\delta_1\ldots\delta_{n-1}1}\epsilon_{\alpha_1\beta_1}\epsilon_{\alpha_2\beta_2}\ldots\epsilon_{\alpha_{n-1}\beta_{n-1}}\epsilon_{\gamma_1\delta_1}\epsilon_{\gamma_2\delta_2}\ldots\epsilon_{\gamma_{n-1}\delta_{n-1}}\right.$$
$$+\sum a_{\alpha_1\ldots\alpha_{n-1}1}\ a_{\beta_1\ldots\beta_{n-1}1}\ a_{\gamma_1\ldots\gamma_{n-1}0}\ a_{\delta_1\ldots\delta_{n-1}0}\epsilon_{\alpha_1\beta_1}\epsilon_{\alpha_2\beta_2}\ldots\epsilon_{\alpha_{n-1}\beta_{n-1}}\epsilon_{\gamma_1\delta_1}\epsilon_{\gamma_2\delta_2}\ldots\epsilon_{\gamma_{n-1}\delta_{n-1}}$$
$$-\sum a_{\alpha_1\ldots\alpha_{n-1}0}\ a_{\beta_1\ldots\beta_{n-1}1}\ a_{\gamma_1\ldots\gamma_{n-1}1}\ a_{\delta_1\ldots\delta_{n-1}0}\epsilon_{\alpha_1\beta_1}\epsilon_{\alpha_2\beta_2}\ldots\epsilon_{\alpha_{n-1}\beta_{n-1}}\epsilon_{\gamma_1\delta_1}\epsilon_{\gamma_2\delta_2}\ldots\epsilon_{\gamma_{n-1}\delta_{n-1}}$$
$$\left.-\sum a_{\alpha_1\ldots\alpha_{n-1}1}\ a_{\beta_1\ldots\beta_{n-1}0}\ a_{\gamma_1\ldots\gamma_{n-1}0}\ a_{\delta_1\ldots\delta_{n-1}1}\epsilon_{\alpha_1\beta_1}\epsilon_{\alpha_2\beta_2}\ldots\epsilon_{\alpha_{n-1}\beta_{n-1}}\epsilon_{\gamma_1\delta_1}\epsilon_{\gamma_2\delta_2}\ldots\epsilon_{\gamma_{n-1}\delta_{n-1}}\right|. \quad (5)$$

Consider some term in the fully expanded version of the first line of the above equation, $a_{\mu_1\ldots\mu_{n-1}0}\ a_{\bar{\mu}_1\ldots\bar{\mu}_{n-1}0}\ a_{\nu_1\ldots\nu_{n-1}1}\ a_{\bar{\nu}_1\ldots\bar{\nu}_{n-1}1}$, where $\bar{\mu}=1$ if $\mu=0$ and $\bar{\mu}=0$ if $\mu=1$. This term can be positive or negative. The expansion of the first line of the above equation also contains the term $a_{\bar{\mu}_1\ldots\bar{\mu}_{n-1}0}\ a_{\mu_1\ldots\mu_{n-1}0}\ a_{\nu_1\ldots\nu_{n-1}1}\ a_{\bar{\nu}_1\ldots\bar{\nu}_{n-1}1}$. For even n the sign of this term will be opposite to the sign of the original term, since the signs of an odd number of ϵ's have been flipped. So the two above terms will add to zero, as will all other terms in the first line of Eq. (5). The second line of Eq. (5) also goes to zero by the same argument. Thus $\tau_{1\ldots n}=|\langle\psi|\tilde{\psi}\rangle|^2$ for all even n.

This equality indicates that the n-tangle is a more natural measure of entanglement than concurrence because, for odd n, $C_{1\ldots n}=0$, while the meaning of the n-tangle is already established for $n=3$ [4]. From Eq. (3) one can determine that the quantity $C^2_{1\ldots n}=\tau_{1\ldots n}$ for even n is invariant under permutations of the qubits, since changing the order of the indices (i.e., the numbering of the Greek letters) is only renaming the indices. This allows us to apply the method used in Ref. [7] to prove that τ_{ABC} is an entanglement monotone to prove that the n-tangle is an entanglement monotone, a property that good measures of entanglement must satisfy [6]. As in Ref. [7] (we explicitly follow their form and proof outline), the invariance of the n-tangle under permutations of the parties lets us consider local positive operator valued measures (POVM's) for one party only. Let A_1 and A_2 be two POVM elements such that $A_1^\dagger A_1+A_2^\dagger A_2=I$, then $A_i=U_iD_iV$, with U_i and V being unitary matrices, and D_i being diagonal matrices with entries (a,b) and $(\sqrt{1-a^2},\sqrt{1-b^2})$, respectively. For some initial state $|\psi\rangle$ let $|\hat{\phi}_i\rangle=A_i|\psi\rangle$ be the subnormalized states obtained after application of the POVM. Let $|\phi_i\rangle=|\hat{\phi}_i\rangle/\sqrt{p_i}$, $p_i=\langle\hat{\phi}_i|\hat{\phi}_i\rangle$. Then

$$\langle\tau\rangle=p_1\tau(\phi_1)+p_2\tau(\phi_2). \tag{6}$$

Since the n-tangle is invariant under local unitarities [7] $\tau(U_iD_iV\psi)=\tau(D_iV\psi)$. Now, noting that every term of Eq. (2) contains two a's with subscripts starting with zeros and two a's with subscripts starting with 1's, and that every term is quartic with respect to the a's, it can be shown that

$$\tau(\phi_1)=\frac{a^2b^2}{p_1^2}\tau(\psi),\qquad \tau(\phi_2)=\frac{(1-a^2)^2(1-b^2)^2}{p_2^2}\tau(\psi). \tag{7}$$

Defining P_0 to be the sum of the squared magnitudes of the first 2^{n-1} components of $|\psi\rangle$ in the standard basis, and P_1 to be the sum of the squared magnitudes of the last 2^{n-1} components of $|\psi\rangle$, we can say that

$$p_1=a^2P_0+b^2P_1 \quad\text{and}\quad p_2=(1-a^2)P_0+(1-b^2)P_1. \tag{8}$$

Combining Eqs. (6)–(8) with the fact that $P_0+P_1=1$, some algebra shows that $\langle\tau\rangle/\tau(\psi)\le 1$, thus proving that the n-tangle is an entanglement monotone.

Some examples provide further support for the n-tangle being a measure of some type of n-party entanglement. An n-qubit (Schrödinger) CAT state, $(|0^{\otimes n}\rangle+|1^{\otimes n}\rangle)/\sqrt{2}$, is a state with entirely n-way entanglement; measuring any one of the qubits in the standard basis determines the value of all of the other qubits; however, if one of the qubits is traced out, the remaining qubits are unentangled. For these states, the n-tangle is 1; all terms in Eq. (2) are 0 for an n-CAT state except for when $\alpha_i=0$, $\beta_i=1$, $\gamma_i=1$, and $\delta_i=0$, or $\alpha_i=1$, $\beta_i=0$, $\gamma_i=0$, and $\delta_i=1$, so $\tau_{1\ldots n}(|\mathrm{CAT}\rangle)=2|-1/4+-1/4|=1$. Another interesting set of states are the n-qubit W states [7] $(|0\ldots01\rangle+|0\ldots010\rangle+\cdots+|10\ldots\rangle)/\sqrt{n}$. For these states, the equality [4]

$$C^2_{12}+C^2_{13}+\cdots+C^2_{1n}=C^2_{1(23\ldots n)} \tag{9}$$

holds; thus tracing out all but two of the qubits leaves the two remaining qubits partially entangled. Note that $C^2_{1(23\ldots n)}\neq 0$, and that the W states are symmetric. Note that the ϵ's assure that all terms in Eq. (2) are 0 for W states, and hence the n-tangle is zero for W states (except for $n=2$, where $\tau=1$, since τ measures two-way entanglement in this case). From the above examples, it is tempting to hypothesize that the n-tangle is a measure of n-way entanglement, but a counterexample shows otherwise: Consider the four-qubit pure state that is the tensor product of two singlet states. A simple calculation shows that the 4-tangle has a value of 1 for this state. If the 4-tangle measured four-way entanglement, its value should have been 0, since this state has no entanglement between the pairs of entangled qubits. Thus, while the n-tangle appears to be related to some kind of multipartide entanglement, it is not by itself a measure of n-way entanglement.

III. MIXED-STATE GENERALIZATION OF n-TANGLE

We would like to have a mixed-state generalization of the n-tangle. Such a quantity would enable us to classify and quantify even more types of entanglement. For example, a four-qubit pure state would have six values of the mixed-state 2-tangle between each of the pairs of qubits, four values for the mixed-state 3-tangle between each set of three qubits, and a value for the 4-tangle. We suggest defining, for an n-qubit mixed state ρ, $\tau^{min}(\rho)$ to be the minimum of $\Sigma_i p_i\tau(\psi_i)$ for all pure-state decompositions of ρ, given by $\rho=\Sigma_i p_i|\psi\rangle\langle\psi|$. This is analogous to the entanglement of formation [8], and in fact the entanglement of formation is a function of $\tau^{min}(\rho_{12})$ for states of two qubits [5]. This definition is also justified by the fact that Eq. (24) of Ref. [4] can now be rewritten as

$$C^2_{A(BC)}=\tau^{min}_{AB}+\tau^{min}_{AC}+\tau_{ABC}. \tag{10}$$

That is, for two qubits, $\tau^{min}_{12}(\rho)$ already has physical significance, so it appears to be a natural way to define a mixed state τ. Now, in Ref. [5], Wootters presented a proof that $C_{min}(\rho)=\max\{0,\lambda_1-\lambda_2-\lambda_3-\lambda_4\}$ where λ_i is the square root of the ith eigenvalue, in decreasing order, of $\rho\tilde{\rho}$. This proof is generalizable to show that $C_{min}(\rho)=max\{0,\lambda_1-\lambda_2\ldots-\lambda_n\}$ for an n-qubit system, and therefore $\tau^{min}(\rho)=C^2_{min}(\rho)=[\max\{0,\lambda_1-\lambda_2\ldots-\lambda_n\}]^2$. This result is also a subset of a more general proof by Uhlmann [18].

A large number of doubts remain about the meaning of the n-tangle. In particular, we would like to have a physically meaningful definition of n -way entanglement, so that we could compare the n-tangle and other multipartide entanglement measures with meaningful values. It seems likely that the n-tangle, in combination with other multipartide entanglement measures (most likely the n-tangles of smaller subsystems within a given state), will be related to a multipartide generalization of the two-qubit entanglement E related to the Shannon entropy. Unfortunately, no such generalization of entanglement is obvious. If a formula for $\tau^{min}_{1\ldots n}$ for $n=3$ could be found, it might be possible to prove statements analogous to Eq. (10) which would lend more legitimacy to the n-tangle. We would also like to have a generalization of the n-tangle for states with subsystems larger than qubits.

IV. DISCUSSION

In summary, we have proposed a potential measure of a type of n-partide entanglement of pure and mixed states: for pure states the n-tangle, and for mixed states the related $\tau^{min}_{1\ldots n}$. These measures show many signs of being useful ways to quantify a type of multipartide entanglement. For even n, the n-tangle and $\tau^{min}_{1\ldots n}$ are equal to the square of a generalization of pure- and mixed-state concurrence, and the n-tangle is also an entanglement monotone. The n-tangle has values of 1 for n-CAT states and values of 0 for W states where $n>2$, but has a value of 1 for a product state of two singlets. Hopefully these measures will further our understanding of multipartide entanglement. In particular, further exploration of their mixed-state forms may lead to the discovery of relationships between different types of entanglement within a particular system.

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