A. 
\[ \vec{v} + \vec{w} = \langle 3, 4, 2, 3 \rangle. \]
\[ 4\vec{v} = \langle 8, 4, -4, 4 \rangle. \]
\[ \vec{v} \cdot \vec{w} = 2 + 3 - 3 + 2 = 4. \]
\[ \text{proj}_{\vec{w}} \vec{v} = \frac{\vec{v} \cdot \vec{w}}{\vec{w} \cdot \vec{w}} \vec{w} = \frac{4}{23} \langle 1, 3, 3, 2 \rangle. \]

B. If \( f = -GM(x^2 + y^2 + z^2)^{-1/2} \), then
\[ f_x = -GM(-1/2)(x^2 + y^2 + z^2)^{-3/2}2x = GM(x^2 + y^2 + z^2)^{-3/2}x. \]
The other partials \( f_y \) and \( f_z \) are similar, with the consequence that
\[ -\nabla f = -GM(x^2 + y^2 + z^2)^{-3/2} \vec{x} = \frac{GM}{|\vec{x}|^2} \cdot -\vec{x}. \]
Here, \(-\vec{x}/|\vec{x}|\) is the unit vector pointing from \( \vec{x} \) toward the origin. So the acceleration has magnitude \( GM/|\vec{x}|^2 \) in that direction.

C1. Let \( \vec{r}(t) = (t, f(t), 0) \). Then \( \vec{r}'(t) = (1, f'(t), 0) \), \( \vec{r}''(t) = (0, f''(t), 0) \), and
\[ \kappa = \frac{|\vec{r}' \times \vec{r}''|}{|\vec{r}'|^3} = \frac{|(0, 0, f''(t))|}{|(1, f'(t), 0)|^3} = \frac{|f''(t)|}{(1 + (f'(t))^2)^{3/2}}. \]

C2. Notice that the curve lies entirely in the \( x-y \)-plane. Hence its unit tangent vector \( \vec{T} \) is always in that plane, the derivative of \( \vec{T} \) is always in that plane, and so is the unit normal \( \vec{N} = \vec{T}'/|\vec{T}'| \). (This result matches our intuition, that \( \vec{N} \) always points “into the turn” of the curve. For that to be true of a plane curve, \( \vec{N} \) must lie in the plane.) Thus the unit binormal \( \vec{B} = \vec{T} \times \vec{N} \) is \( \langle 0, 0, \pm 1 \rangle \) wherever it is defined, and \( d\vec{B}/ds = \vec{0} \) wherever it exists. The torsion \( \tau \), being defined by \( d\vec{B}/ds = -\tau \vec{N} \), must be 0 everywhere it exists.

D. We first express the locations of the two cities in spherical coordinates on a sphere of radius 1. Johannesburg is at \( \phi = 116^\circ, \theta = 28^\circ \), while Phnom Penh is at \( \phi = 78^\circ, \theta = 105^\circ \). Converting to Cartesian coordinates, we have Johannesburg at \( \vec{j} = (\sin 116^\circ \cos 28^\circ, \sin 116^\circ \sin 28^\circ, \cos 116^\circ) \) and Phnom Penh at \( \vec{p} = (\sin 78^\circ \cos 105^\circ, \sin 78^\circ \sin 105^\circ, \cos 78^\circ) \). Then
\[ \vec{j} \cdot \vec{p} = |\vec{j}||\vec{p}| \cos \alpha = \cos \alpha, \]
where \( \alpha \) is the central angle between the two vectors. Finally, the distance along the sphere in km is 6371\( \alpha \), which equals
\[ 6371 \arccos(\sin 116^\circ \cos 28^\circ \sin 78^\circ \cos 105^\circ + \sin 116^\circ \sin 28^\circ \sin 78^\circ \sin 105^\circ + \cos 116^\circ \cos 78^\circ). \]
E. In two dimensions we have polar coordinates

\[
x = \cos \theta, \\
y = \sin \theta
\]
on the unit circle, and in three dimensions we have spherical coordinates

\[
x = \sin \phi \cos \theta, \\
y = \sin \phi \sin \theta, \\
z = \cos \phi
\]
on the unit sphere. Continuing this pattern, we try

\[
x = \sin \psi \sin \phi \cos \theta, \\
y = \sin \psi \sin \phi \sin \theta, \\
z = \sin \psi \cos \phi, \\
w = \cos \psi
\]
on the unit hypersphere in four dimensions. One can check that \(x^2 + y^2 + z^2 + w^2 = 1\), as desired. One can also check that all regions of the sphere are covered, although that is harder to do. By the way, here is another answer:

\[
x = \sin \phi \cos \theta, \\
y = \sin \phi \sin \theta, \\
z = \cos \phi \cos \psi, \\
w = \cos \phi \sin \psi.
\]

F1. Let \(f(x, y) = \frac{x^3y}{x^2 + y^2}\). Along the line \(y = mx\),

\[
f = \frac{mx^4}{x^6 + m^2x^2} = \frac{mx^2}{x^4 + m^2}.
\]
If \(m = 0\), then \(f = 0/x^4 = 0 \to 0\) as \(x \to 0\). If \(m \neq 0\), then \(f \to 0/m^2 = 0\) as \(x \to 0\). Finally, along the vertical line \(x = 0\), \(f = 0/m^2 = 0 \to 0\) as \(x \to 0\). Thus \(f\) goes to 0 along every line through the origin.

F2. Along the curve \(y = x^3\),

\[
f = \frac{x^6}{x^6 + x^6} = 1/2.
\]
Thus \(f \to 1/2\) along this curve. Because this apparent limit disagrees with those found in part F1, we conclude that \(\lim_{(x,y) \to (0,0)} f\) does not exist.
G. No, there does not exist a function $f(x, y)$ such that $\nabla f = \langle x^2 \cos(x^3y^3), y^2 \sin(x^3y^3) \rangle$. For suppose such an $f$ did exist. Then

$$f_x = x^2 \cos(x^3 y^3) \Rightarrow f_{xy} = -x^2 \sin(3x^2 y^2),$$

while

$$f_y = y^2 \sin(x^3 y^3) \Rightarrow f_{yx} = y^2 \cos(3x^2 y^3).$$

Because $f_{xy}$ and $f_{yx}$ are compositions of continuous functions, they are continuous, and by Clairaut’s theorem they must agree. But they do not. Hence $f$ cannot exist.

H. In Cartesian coordinates, the circle of radius $R$ centered at $(R, 0)$ is

$$(x - R)^2 + y^2 = R^2.$$ 

Substituting $x = r \cos \theta$ and $y = r \sin \theta$ yields

$$(r \cos \theta - R)^2 + (r \sin \theta)^2 = R^2,$$

which is equivalent to

$$r^2 \cos^2 \theta - 2r R \cos \theta + R^2 + r^2 \sin^2 \theta = R^2,$$

which simplifies to

$$r - 2R \cos \theta = 0.$$