A. Assume for the sake of contradiction that $A$ is regular. Then by the pumping lemma there exists a pumping length $p$, such that for the string $b^p a^{p+1} \in A$, there exist strings $x$, $y$, $z$ satisfying $b^p a^{p+1} = xyz$, $|xy| \leq p$, $|y| \geq 1$, and $xy^iz \in A$ for all $i \geq 0$. But $y = b^k$ for some $k$ satisfying $1 \leq k \leq p$. So $xyyz = b^{p+k} a^{p+1}$ is not a string in $A$. This contradiction shows that $A$ is not regular.

For a CFG, I think that $S \to a|aS|aS|bSa|abS|baS$ works. (Checking these is not easy. In grading, I gave full credit to a CFG, if I couldn’t quickly detect a defect in it.) For a PDA, simply take any CFG and apply our algorithm for converting a CFG to a PDA.

B. The language is regular. There is a simple three-state DFA for it. I’ll omit the drawing.

C. Let $L$ match any one of the 52 letters, and let $A$ match any one of the 62 alphanumeric characters. Then a regular expression $H$ for hostnames is

$$(AA^*.)^* AA^*.LL^*$$

Let $C$ match any letter, digit, period, or underscore. Then a regular expression $P$ for paths is

$$/(CC^*/)*/(\epsilon \cup CC^*)$$

Let $D$ match any of the ten digits. Then a regular expression for URLs is

$$LL^*://H(\epsilon \cup DD^*)(\epsilon \cup P \cup P\#CC^*)$$

D. Let $A = \{a^ib^j : \text{exactly one of } i, j \text{ is a multiple of } 2, \text{ and exactly one of } i, j \text{ is a multiple of } 3\} \subseteq \{a,b\}^*$. Prove that $A$ is regular, or prove that $A$ is not regular.

It is easy to construct DFAs that match each of these languages:

- $A_{2i} = \{a^ib^j : 2 \text{ divides } i\}$.
- $A_{2j} = \{a^ib^j : 2 \text{ does not divide } i\}$.
- $A_{3i} = \{a^ib^j : 3 \text{ divides } i\}$.
- $A_{3j} = \{a^ib^j : 3 \text{ does not divide } i\}$.
The language $A$ is a union of intersections of these languages:

$$A = \left( A_{2i} \cap A_{2j} \cap A_{3i} \cap A_{3j} \right) \cup \left( A_{2i} \cap A_{2j} \cap A_{3i} \cap A_{3j} \right) \cup \left( A_{2i} \cap A_{2j} \cap A_{3i} \cap A_{3j} \right).$$

Because each of the eight languages is regular, and regular languages are closed under intersection and union, $A$ must also be regular.

E. Let $A$ be an infinite subset of $\{a^n b^n : n \geq 0\}$. Assume for the sake of contradiction that $A$ is regular. By the pumping lemma, there exists a pumping length $p$ for $A$. Because $A$ is infinite, it must contain at least one string $w$ of length at least $2p$. This $w = a^q b^q$ for some $q \geq p$. By the pumping lemma, there exist $x, y, z$ such that $w = xyz$, $|xy| \leq p$, $|y| > 0$, and $xy^i z \in A$ for all $i \geq 0$. Clearly $y = a^k$ for some $1 \leq k \leq p$, so $xz = a^{q-k} b^q$ is not in $A$ after all. This contradiction shows that $A$ is not regular. Thus no infinite subset of $\{a^n b^n : n \geq 0\}$ is regular.