1. Differentiate the following functions. Do NOT simplify.

A. \( y = 5x^6 - 2x^3 + 3x + 1 \).
\[
\frac{dy}{dx} = 30x^5 - 6x^2 + 3.
\]

B. \( s(t) = \frac{t}{\cos t} \).
\[
\frac{ds}{dt} = \frac{\cos t + t \sin t}{\cos^2 t}.
\]

C. \( p = 4^r \cdot (3r - 1) \).
\[
\frac{dp}{dr} = 4^r \ln 4 \cdot (3r - 1) + 4^r \cdot 3.
\]

D. \( y = \left(2e^{5t} + \sin \left(\sqrt{t}\right)\right)^7 \).
\[
\frac{dy}{dt} = 7 \left(2e^{5t} + \sin \left(\sqrt{t}\right)\right)^6 \cdot \left(2e^{5t} \cdot 5 + \cos \left(\sqrt{t}\right) \cdot \frac{1}{2} t^{-1/2}\right).
\]

2. The Furtwängler Glacier on Mount Kilimanjaro is a giant sheet of ice. Suppose, for simplicity, that it is rectangular of length 600 m and width 100 m. Due to warming climate, its length and width are both shrinking by 5 m per year. Its thickness is a constant 6 m. How fast is the volume of the glacier changing? Simplify your answer, and include units.

Since the volume \( v \) is the product of the thickness 6, length \( \ell \), and width \( w \), we have \( v = 6\ell w \).

Let \( t \) be time in years. Then \( \frac{dt}{dt} = \frac{dw}{dt} = -5 \), and
\[
\frac{dv}{dt} = \frac{d}{dt}(6\ell w) = 6 \left(\frac{d\ell}{dt} w + \ell \frac{dw}{dt}\right) = 6(-5 \cdot 100 + 600 \cdot -5) = -30 \cdot 700 = -21000.
\]
Thus the glacier is losing 21000 m\(^3\) of volume per year.

3. Suppose that \( x = f(t) = \cos(2t) + \sin(3t) \) is the position of a particle at time \( t \). Compute the particle’s acceleration. Simplify and clearly mark your answer.

The velocity is \( f'(t) = -2\sin(2t) + 3\cos(3t) \). The acceleration is \( f''(t) = -4\cos(2t) - 9\sin(3t) \).

4. Recall that Newton’s law of cooling is expressed by the differential equation \( \frac{dy}{dt} = k(A - y) \).

A. Explain in words the physical meaning of these five quantities:
t is time.
y is the temperature of the cooling (or warming) body.
A is the ambient temperature of the environment, assumed to be constant.
k is a constant of proportionality with units 1 / time.
\( \frac{dy}{dt} \) is the rate of change of the body temperature with respect to time.

B. We used \( k = 0.9 \) in the Santa-cooling problem in class. If we had used a lesser (but still positive) value of \( k \), such as 0.5, would Santa cool more slowly or more rapidly? Explain.

If \( k \) were made smaller, then \( \frac{dy}{dt} \) would become smaller, meaning that the rate of change of Santa’s temperature would be smaller. So he would cool more slowly.

C. The value \( k = 0.9 \) was given to us by the police but not explained further. In words, describe some circumstances of Santa’s death that might influence the police estimate for \( k \).

What circumstances might affect the rate at which Santa cools (other than the ambient temperature, which is already described by \( A \))? It comes down to various forms of insulation: body fat, clothing, whether he was indoors (in a house or car) or outdoors, how much wind there was, etc.

5. Show that \( \lim_{h \to 0} h \sin \left( \frac{1}{h} \right) = 0 \).

Since \( \sin x \) is always between \(-1\) and \(1\), so is \( \sin(1/x) \). It follows that \( x \sin(1/x) \) is always between \(-|x|\) and \(|x|\). (To convince yourself, you can plot \( y = x, y = -x, \) and \( y = x \sin(1/x) \) all on one graph.) Thus for all \( h \neq 0 \),

\[-|h| \leq h \sin(1/h) \leq |h|.
\]

Now \(-|h|\) and \(|h|\) both have limit 0 as \( h \to 0 \), and \( h \sin(1/h) \) is trapped between them, so the Squeeze Theorem says that \( h \sin(1/h) \) must also have limit 0 as \( h \to 0 \). (This problem was taken from our homework; see Problem 6 below.)

6. Compute the derivative, at \( x = 0 \), of \( y = f(x) = \begin{cases} x^2 \sin \left( \frac{1}{x} \right) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases} \)

We use the definition of the derivative:

\[
f'(0) = \lim_{h \to 0} \frac{f(0 + h) - f(0)}{h} \\
= \lim_{h \to 0} \frac{f(h) - f(0)}{h} \\
= \lim_{h \to 0} \frac{h^2 \sin \left( \frac{1}{h} \right) - 0}{h} \\
= \lim_{h \to 0} h \sin \left( \frac{1}{h} \right) \\
= 0,
\]

by Problem 5 above. (This problem was taken from our homework, namely 2.7 #52.)
7. The table below shows the number $s$ of web sites in existence, for each year $t$ in a 10-year period. I wish to model $s$ as a function of $t$. When I plot the data on a semilog plot, I get a line of slope 1.43 and intercept 2.31. What then is the function $s(t)$? Simplify your answer.

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<td>16400000</td>
</tr>
</tbody>
</table>

The semilog plot relates $t$ to $\ln s$, so the equation of the line is $\ln s = 1.43t + 2.31$. Exponentiating both sides of that equation, we get

$$s = e^{1.43t + 2.31} = e^{2.31}e^{1.43t}.$$ 

8. Graph $y = e^{2x} - 4$ as precisely as you can (including the correct scale, intercepts, etc.).

The graph is like that of $y = e^x$, but compressed by a factor of 2 horizontally and shifted down 4. The $y$-intercept is at $(0, -3)$. The $x$-intercept is where

$$e^{2x} - 4 = 0$$
$$\Leftrightarrow (e^x)^2 = 4$$
$$\Leftrightarrow e^x = \pm 2$$
$$\Leftrightarrow e^x = 2$$
$$\Leftrightarrow x = \ln 2.$$ 

So it’s at $(\ln 2, 0)$. I’ll omit the graph from these solutions; you can graph it for yourself using Mathematica or a graphing calculator.